THESIS

on

# THE STUDY OF HIGHER TRANSCENDENTAL FUNCTIONS OF SEVERAL VARIABLES

PRESENTED

by

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for the degree of

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# CERTIFICATE

This is to certify that Mr. H.C. Yadava of D.V. Postgraduate College Orai actually carried out the work described in this thesis, under my supervision at this College. He has put the required attendance in the department during the period of his investigation.

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PREFACE

I
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I

The present work is the out come of the research carried out by me in the field of Higher Transcendental Functions of Several Variables of Mathematical Physics, at the Department of Mathematics of D.V. Postgraduate College Orai-285001, U.P. India.

This thesis consists of eight chapters, each divided into several sections (progressively numbered 1.1, 1.2,...). The formulae are numbered progressively within each section. For example, (7.3.8) denotes the 8th formula of 3rd section in Chapter VII. References are given in alphabetical order at the end of each chapter.

This work was started since first January 1979 under the able supervision of Dr. R.C. Singh Chandel, M.Sc., Ph.D. of D.V. Postgraduate College, Orai. Since then I have been regularly receiving valuable suggestions and guidance. I wish to record my deepest and sincerest feeling of gratitude to Dr. Chandel, but for whose worthy guidance, it would not have been possible for me to accomplish my purpose.

I also wish to record my sincere feeling of thanks to the Principal, D.V. Postgraduate College Orai for the facilities that he provided me during the period of my research work.

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# A LIST OF RESEARCH PUBLICATIONS

1. A note on a generating function for certain polynomial systems, Ranchi Uni. Math. Jour. 10 (1979), 61-66.
2. Some generating Functions for certain polynomial systems of several variables, Proc. Nat. Acad. Sci. India 51(A), II, (1981), 133-138.
3. Use of multiple hypergeometric functions of Srivastava and Daoust in cooling of a heated cylinder, Indian Jour. of Math. (Communicated).
4. Heat conduction and Multiple Hypergeometric functions of Srivastava and Daoust, Indian J. Pure and Appl. Math. (Communicated).
5. Operational Representation of certain generalized hypergeometric functions in several variables, Indian J. Math. (Communicated).
6. Additional Application of Binomial Analog of Srivastava theorem. Indian J. Math. (Communicated).
7. Application of Srivastava theorem, Indian J. Pure and Appl. Math. (Communicated).
8. A Binomial Analog of Srivastava theorem, Indian J. Pure and Appl. Math. (Communicated).

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CHAPTER I

I
I
I

## A BRIEF HISTORICAL REVIEW

In this chapter we propose to give a brief historical account of some of the work done in the field of 'Higher Transcendental Functions of Several Variables'. No attempt has been made to give a comprehensive review of the entire literature on the subject but only those aspects, which have a direct bearing on my work done in the present thesis, have been dealt with in some details.

1.1 Higher transcendental functions. An equation of the form

(1.1.1) $$p_o(x)\ W^n + p_1(x)\ W^{n-1} + \ldots + p_n(x) = 0,$$

where $p_o(x)$, $p_1(x), \ldots, p_n(x)$ are polynomial expressions having integral coefficients, e.g.

$$p_o(x) = 5x^4 + 3x^3 + 7x^2 - 8x+3,$$

$$p_1(x) = 11x^9-2$$

is called algebraic equation. The roots of the above equation

(1.1.2) $$\omega = f(x)$$

are called algebraic functions. The functions which are not roots of algebraic equations are called transcendental functions. Logarithmic functions, exponential functions, trigonometric

functions etc. are the examples of transcendental functions.

Transcendental functions are generally solution of differential equations or they have integral representations, Transcendental functions such as Beta functions, Gamma functions, Bessel functions, E, G, H functions, all polynomials etc., which are of complicated nature are known as Higher Transcendental Functions.

In the study of higher transcendental functions, if we are not concerned with their general properties, but only with the properties of the functions which occur in special problems, they are called 'special functions'. Moreover, it is a matter of opinion or convention. According to Harry Bateman (1882-1946) any function which has received individual attention at least in one research paper, may be attributed to Special Function.

Special functions of mathematical physics arise in the solution of partial differential equations governing the behaviour of certain physical quantities. All these functions are H.T.Fs. The equations which occur frequently in pure and applied sciences are

(1.1.3) $$\nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$ (Wave equation),

(1.1.4) $$\nabla^2 \Phi = 0$$ (Laplace equation),

(1.1.5) $$\nabla^2 \Phi = \frac{1}{K} \frac{\partial^2 \Phi}{\partial t^2}$$ (Diffusion equation).

The operator $\nabla^2$, read as 'nabla squared' , is defined as in the following cartesian form :

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} . \tag{1.1.6}$$

In each of the above equations t denotes time variable, c and K physical quantities which are generally constants and the function $\Phi$ has to be determined.

Its physical meaning depends upon the nature of the problem. The equation (1.1.3) arises in the problems which involve the phenomenon of wave motion and which occur in electromagnetism, acoustics, elasticity, hydrodynamics, etc. Equation (1.1.4) arises in potential problem, which occur in many branches of pure and applied sciences, hydrodynamics, electrostatics, steady flow of heat and current, gravitation and elasticity. Equation (1.1.5) reduces to (1.1.4) when $\Phi$ is time independent. In its general form it occurs in the theory of flow of heat, the skin effect for an alternating current in a conductor, in the theory of the transmission line and in certain diffusion problems. A wide range of physical problems are represented by the equations (1.1.3), (1.1.4) and (1.1.5).

There are various methods of solving these equations one of the important method of separation of variables is generally employed to solve them. The study of differential equations describing the physical situation and consistent

with the boundary conditions leads us to the higher transcendental functions of mathematical physics. Here we shall discuss some higher transcendental functions, particularly, the polynomials and their generalizations. We shall also discuss the hypergeometric functions in one, two, three, four and several variables.

1.2 Legendre function. Higher transcendental functions were first introduced towards the end of eighteenth century in the solution of the problems of dynamical Astronomy and Mathematical Physics. In 1782, Laplace introduced the potential theorem. Legendre (1782 or earlier) investigated the expansion of potential function in the form of an infinite series and was thus led to the discovery of functions now known as Legendre coefficients.

Thomson and Tait in their well known "Natural Philosophy" (1879) defined spherical harmonics as follows :

Any solution $V_n$ of laplace equation $\nabla^2 \Phi = 0$, which is homogeneous of degree n in x,y,z is called a "solid spherical harmonics of degree n". The degree n may be any number and the function need not be rational.

If x,y,z are expressed in terms of polar coordinates $(r,\theta,\Phi)$ the solid spherical harmonic of degree n assumes the form $r^n f_n(\theta,\Phi)$.The function $f_n(\theta,\Phi)$ is called a "surface spherical harmonic of degree n". Laplace equation possesses

solutions of the form $\left.\begin{matrix} r^n \\ r^{n-1} \end{matrix}\right\} e^{im\Phi}$ $\Theta(\mu)$, where $\Theta(\mu)$ satisfies the ordinary differential equation

$$(1.2.1) \qquad (1-\mu^2)\,\frac{d^2\Theta}{d\mu^2} - 2\mu\,\frac{d\Theta}{d\mu} + \{n(n+1) - \frac{m^2}{1-\mu^2}\}\;\Theta = 0.$$

The above equation is called associated Legendre equation, $\mu$ is restricted to be real and to lie in the interval (-1,1).

1.3 Hermite polynomials. Hermite polynomials, first of all, were discussed by laplace in his two works 'Treatise on Celestial Mechanics' (18 05) [39] and 'theory of probability' (1820) [40]. The systematic study of these polynomials was made by Ch. Hermite [32] in 1864.

Hermite polynomials occur in case of the motion of a point mass in a field of force. Schrödinger [46] showed that a free particle which is represented by a wave function $\Psi(r,t)$, r being the position vector of the particle, satisfies the following differential equation :

$$(1.3.1) \qquad i\,h\,\frac{\partial\Psi}{\partial t} = -\,\frac{h^2}{2m}\;\nabla^2\Psi\;,$$

h being an universal constant. If the particle includes the effect of the external forces such as electrostatic, gravitational, possibly nuclear which can be combined into a single force F, that is, derivable from the potential energy V, the above equation may be generalized into

$$(1.3.2) \qquad i\, h \frac{\partial \Psi}{\partial t} = - \frac{h^2}{2m} \nabla^2 \Psi + V(r,t)\, \Psi \, .$$

If the potential energy is independent of the time and $\Psi(r,t) = u(r)\ f(t)$, the equation may be separated into

$$(1.3.3) \qquad \left[ - \frac{h^2}{2m} \nabla^2 + V(r) \right] u(r) = E.u(r),$$

E being the separation constant.

Further, one dimensional motion of a point mass attracted to a fixed centre by a force proportional to the displacement from that centre, provides one of the fundamental problem of classical dynamics. The force $F = -Kx$ can be represented by potential energy $V(x) = \frac{Kx^2}{2}$, so that Schrödinger's equation in one dimension may be written in the form :

$$(1.3.4) \qquad - \frac{h^2}{2m} \frac{d^2 u}{dx^2} + \frac{1}{2} Kx^2 = E.u.$$

Substituting $\xi = \alpha x$, the above equation becomes

$$\frac{d^2 u}{d\xi^2} + (\lambda - \xi^2)\, u = 0.$$

We can find an exact solution of the above equation in the form :

$$(1.3.5) \qquad u(\xi) = H(\xi)\, e^{-\xi^2/2},$$

where $H(\xi)$ is a polynomial of finite order in $\xi$ .

This assumption, on substitution into one dimensional equation leads to the differential equation

$$H''(\xi) - 2\xi H'(\xi) + (\lambda-1) H(\xi) = 0.$$

In order to find the solution, choose $\lambda = 2n+1$, so that

$$H_n''(\xi) - 2\xi H_n'(\xi) + 2n H_n(\xi) = 0. \quad (1.3.6)$$

The function $H_n(\xi)$ is called Hermite polynomial of degree n.

1.4. Laguerre polynomials. E. de Laguerre [38] introduced Laguerre polynomials in 1879. These polynomials also occur in an unedited manuscript of Abel [1]. In physical problems these polynomials occur in case of the motion of two particles (nucleus and electron) that are attracted to each other by a force that depends only on the distance between them.

The potential energy $V(r) = -\frac{Ze^2}{r}$, which represents the attractive Coulomb interaction between an atomic nucleus of positive charge +Ze and an electron of negative charge −e, provides a wave equation. The Schrödinger wave equation describes the motion of a single particle in an external field. Now, however, we are interested in the motion of the two particles (nucleus and electron). The differential equation for the energy characteristic state in this case is

$$\frac{1}{2m_1}\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial z_1^2}\right) + \frac{1}{2m_2}\left(\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial y_2^2} + \frac{\partial^2 u}{\partial z_2^2}\right) + \frac{1}{h^2}\left[E_0 - V\{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\}\right] u = 0. \quad (1.4.1)$$

Subscripts 1 and 2 refer to first and second particles.

Expressing the equation in terms of the coordinates of centre of mass and using the method of separation of variables, the above equation gives

$$(1.4.2) \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \left[\frac{2\mu}{h^2}(E-V) - \frac{C}{r^2}\right] R = 0,$$

where C is constant of separation. By suitably adjusting the constants and taking $R = \rho^{l} e^{-\rho/2} V(\rho)$, where $\rho = kr$, the equation finally reduces to

$$(1.4.3) \qquad \frac{d^2V}{d\rho^2} + \left[2\frac{(\ell+1)}{\rho} - 1\right]\frac{dV}{d\rho} + [C - (\ell+1)]\frac{V}{\rho} = 0.$$

The physically acceptable solution of this equation with C = n may be represented in terms of Laguerre polynomials. These polynomials satisfy the following differential equation :

$$(1.4.4) \qquad x \frac{d^2}{dx^2} L_n^{(\alpha)}(x) + (1+\alpha-x)\frac{d}{dx} L_n^{(\alpha)}(x) + n\, L_n^{(\alpha)}(x) = 0.$$

The polynomials stated above are called classical orthogonal polynomials.

1.5. <u>Orthogonal polynomials</u>. If $\{\Phi_n(x)\}$ denotes a sequence of functions with a weight function w(x) in an interval (a,b) which is non-negative there, we may associate the scalar product

$$(1.5.1) \qquad (\Phi_1, \Phi_2) = \int_a^b w(x)\, \Phi_1(x)\, \Phi_2(x)\, dx,$$

which is defined for all functions $\Phi$ for which $w^{\frac{1}{2}}\Phi$ is

quadratically integrable in (a,b). Two functions are said to be orthogonal if their scalar product vanishes.

The functions $\{\Phi_n(x)\}$, thus form an orthogonal system if

(1.5.2) $$(\Phi_h, \Phi_k) = \begin{cases} 0 & \text{if } h \neq k \\ 1 & \text{if } h = k \end{cases}$$

Legendre, Gegenbauer, Jacobi, Hermite and Laguerre polynomials each forms an orthogonal set. These polynomials arise very frequently. They have number of common properties; the following three of which are most important :

(i) $\{\Phi_n'(x)\}$ is a system of orthogonal polynomials,

(ii) $\Phi(x)$ satisfy a differential equation of the form

(1.5.3) $$A(x)\, Y'' + B(x)\, Y' + \lambda_n\, Y = 0,$$

where $A(x)$ and $B(x)$ are independent of n and $\lambda_n$ is independent of x,

(iii) there is generalized Rodrigues' formula

(1.5.4) $$\Phi_n(x) = \frac{1}{K_n\, w(x)} \frac{d^n}{dx^n} \left[ w(x)\, X^n \right],$$

where $K_n$ is constant and X is a polynomial in x whose coefficients are <u>independent</u> of n.

Other important properties of orthogonal polynomials are the self adjoint form of the differential equation :

$$(1.5.5) \qquad \frac{d}{dx}\left[Xw(x)\frac{dY}{dx}\right] + \lambda_n\, w(x)\, Y = 0,$$

and the Christoffel-Darboux formula

$$(1.5.6) \qquad \sum_{r=0}^{n} A_n \Phi_r(x)\, \Phi_r(y) = B_n \frac{\Phi_{n+1}(x)\, \Phi_n(y) - \Phi_n(x)\, \Phi_{n+1}(y)}{x-y} \,.$$

1.6. Non orthogonal polynomials. There are several other hypergeometric polynomials which are not orthogonal called non orthogonal.

In 1936 Bateman [5] was interested in constructing inverse laplace transforms. For this purpose he introduced the polynomials

$$(1.6.1) \qquad Z_n(x) = {}_2F_2(-n,\ n+1;\ 1,1;x).$$

Rice [44] made a considerable study of the polynomials defined by

$$(1.6.2) \qquad H_n(\zeta,p,v) = {}_3F_2(-n,n+1,\zeta;p,1;v)$$

Bateman [4] studied the polynomials

$$(1.6.3) \qquad F_n(Z) = {}_3F_2(-n,n+1,\ \tfrac{1}{2}(Z+1);1,1;1),$$

quite extensivity, and which were generalized by Pasternak in the following way :

$$(1.6.4) \qquad F_n^{(m)}(Z) = F\left[\begin{matrix} -n,\ n+1,\ \frac{1}{2}\ (1+Z+m); \\ 1,\qquad m+1\ ; \end{matrix}\ 1\right]$$

Another polynomial, in which the interest is concentrated on a parameter, is Mittag-Leffer polynomial

(1.6.5) $g_n(Z) = 2Z\ {}_2F_2(1-n,1-Z;2;2).$

Bateman (1940) generalized the above polynomials in the form

(1.6.6) $g_n(Z,r) = \frac{(-r)_n}{n!}\ {}_2F_1\ (-n,Z;-r;2).$

Sister Celine (Fesenmyer [26]) concentrated on the polynomials generated by

(1.6.7) $(1-t)^{-1}\ {}_pF_q\left[\begin{matrix} a_1 \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\ -\frac{4xt}{(1-t)^2}\right]$

$$= \sum_{n=0}^{\infty}\ {}_{p+2}F_{q+2}\left[\begin{matrix} -n, n+1, a_1, \ldots, a_p; \\ 1, \frac{1}{2}, b_1, \ldots, b_q; \end{matrix}\ x\right] t^n$$

Her polynomials include Legendre polynomials, some special Jacobi, Rice's $H_n(\zeta,p,v)$, Bateman's $Z_n(x)$, $F_n(Z)$ and Pasternak's polynomials etc., as special cases.

1.7. Generalization of polynomials. Orthogonal polynomials have been generalized in different directions. For the basis of these generalizations in some cases differential equation is taken in others generating functions, recurrence relations, Rodrigues' formula and so on.

(a) Generating functions and their generalizations. The name 'generating function' was first introduced by Laplace in 1812.

If

$$G(x,t) = \sum_{n=0}^{\infty} f_n(x)\ t^n,$$

where $f_n(x)$ is the function of x only, then $G(x,t)$ is called the generating function of $f_n(x)$. Some of the important classes of generating functions which are generally used in the study of polynomials are listed below :

(i) $$G(2xt-t^2) = \sum_{n=0}^{\infty} g_n(x)\ t^n,$$

in which $G(x)$ has a formal series expansion;

(ii) $$e^t\ \Psi(x,t) = \sum_{n=0}^{\infty} \sigma_n(x)\ t^n,$$

where $\Psi(u)$ has formal power series expansion;

(iii) $$A(t)\ \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} Y_n(x)\ t^n;$$

(iv) $$(1-t)^{-c}\ \Psi\left\{\frac{-4xt}{(1-t)^2}\right\} = \sum_{n=0}^{\infty} f_n(x)\ t^n,$$

where $\Psi(u) = \sum_{n=0}^{\infty} \gamma_n\ u^n,\quad \gamma_0 \neq 0.$

A good generalization of (iii) and (iv) can effectively be given by

(1.7.1) $$A(t)\ F\left(-\frac{r^r\ xt}{(1-t)^r}\right) = \sum_{n=0}^{\infty} f_n(x,r)\ t^n,$$

where $F(x)$ admits Taylor series expansion.

For particular interest Chandel ([13] through [16]) studied the polynomials defined by generating function

$$(1.7.2) \quad (1-t)^{-c} \exp\{-(\frac{r}{1-t})^{r} xt\} = \sum_{n=0}^{\infty} f_n^c(x,r)\, t^n,$$

which can be regarded as the generalization of Laguerre polynomials.

Prompted by this work recently, in order to unify several hitherto considered polynomial systems belonging to (or providing extensions of) the families of classical Jacobi, Hermite, Laguerre polynomials Panda [43] also introduced the polynomials defined by

$$(1.7.3) \quad (1-t)^{-c}\, G\left[\frac{xt^{s}}{(1-t)^{r}}\right] = \sum_{n=0}^{\infty} g_n^c\,(x,r,s)\, t^n,$$

where

$$(1.7.4) \quad G(Z) = \sum_{n=0}^{\infty} \gamma_n\, Z^n, \quad (\gamma_0 \neq 0)$$

and c is an arbitrary parameter. These polynomials include several known polynomials as special cases.

Gould [30] considered a class of generalized Humbert polynomials defined by

$$(1.7.5) \quad (C-mxt+yt^{m})^{p} = \sum_{n=0}^{\infty} P_n\,(m,x,y,p,C)\, t^n,$$

where m is a positive integer and other parameters are unrestricted in general.

On the other hand Srivastava [52] considered a class of generalized Hermite polynomials defined by the generating function

$$(1.7.6) \qquad \sum_{n=0}^{\infty} \gamma_n^{(m)}(x) \frac{t^n}{n!} = G(mxt-t^m),$$

where G(Z) is given by (1.7.4) and m is an arbitrary positive integer.

To unify the study of these three general classes of polynomials defined by (1.7.3), (1.7.5) and (1.7.6), Chandel [18] has recently introduced a class of polynomials defined by the generating function

$$(1.7.7) \qquad (C-mxt+yt^m)^p \, G\left[\frac{r^r \, x \, t^s}{(C-mxt+yt^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} R_n^p \,(m,x,y,r,s,C)\, t^n,$$

where m, s are positive integers, other parameters are unrestricted in general, and G(Z) is given by (1.7.4).

Motivated by this work, in this thesis we introduce a class of polynomials of several variables defined by the generating function

$$(1.7.8) \quad (a_o+a_1 x_1 t+\ldots+a_m x_m t^m)^p \, G\left[\frac{r^r \, x_i \, t^s}{(a_o+a_1 x_1 t+\ldots+a_m x_m t^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} B_{n,m,p,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1 \; , \; x_i \\ \vdots \\ x_m \end{bmatrix} t^n,$$

where $i, r, s, m \geq 1$ are integers $x_1, \ldots, x_m$ are complex numbers and other parameters are unrestricted in general and $G(Z)$ is given by (1.7.4).

Also to unify the study of two general classes of polynomials defined by (1.7.5) and (1.7.6), we shall introduce a class of polynomials defined by the generating function

$$(1.7.9) \qquad G(C-mxt+yt^q) = \sum_{n=0}^{\infty} g_n(m,x,y,z,C)\, t^n,$$

where $G(Z)$ is any function of $Z$, $q$ is an arbitrary and other parameters are unrestricted in general.

(b) Generalization of Rodrigues' formula. Rodrigues' formula for Legendre polynomials is

$$(1.7.10) \qquad P_n(x) = \frac{1}{2^n\, n!} \frac{d^n}{dx^n} (x^2-1)^n.$$

Jacobi (1859) was the first, who generalized above formula and introduced Jacobi polynomials, defined by

$$(1.7.11) \quad P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n\, 2!} (1-x)^{-\alpha}(1+x)^{-\beta} \frac{d^n}{dx^n}\{(1-x)^{\alpha+n}(1+x)^{\beta+n}\}.$$

Appell [2] has generalized the Legendre function as

$$(1.7.12) \qquad \frac{d^n}{dx^n}[x^n(1-x^2)^n],$$

while Ghosh [28] has generalized it in the following way :

$$(1.7.13) \qquad \frac{d^n}{dx^n}[x^\mu (\tfrac{1}{x} - x)^\nu],$$

where $\mu$, $\nu$ are constants.

A different type of generalization was made by Kharadze [33], who introduced for the first line the operator $D_k^k$. He defined the Legendre polynomials as

$$(1.7.14) \qquad Q_{mk}(x) = \frac{1}{k^{2m}(2m)!} D_k^{mk} (x^k-1)^{2m},$$

and

$$(1.7.15) \qquad Q_{mk+1}(x) = \frac{1}{k^{2m+1}(2m+1)!} D_k^{mk+k-1} (x^k-1)^{2m+1},$$

where $k \geq 2$ is a fixed positive number. $D_k^k = \frac{d}{dx} \frac{1}{x^{k-2}} \frac{d}{dx}$, $D_k^{k-1} = \frac{1}{x^{k-2}} \frac{d}{dx}$ and $D_k^{mk}$ means the operator $D_k^k$ repeats m times. Further generalization has been made by Sharma [47] in introducing the generalized associated Legendre functions and generalized ultraspherical polynomials. Srivastava [50] has given a generalization of Ghosh function and he has defined the function by

$$(1.7.16) \qquad Y = D_k^{mk} \{x^{\mu} (1-x^k)^{\nu}\},$$

where $\mu$, $\nu$ are constants.

In a way to generalize Laguerre and Hermite polynomials, Gould and Hopper [29] have introduced a function

$$(1.7.17) \qquad H_n^r(x,r,p) = (-1)^n x^{-\alpha} e^{px^r} \frac{d^n}{dx^n} (x^{\alpha} e^{-px^r}),$$

and to generalize Laguerre and Humbert polynomials Singh and Srivastava [48] (Also see Chatterjea [10]) have defined

the polynomials by Rodrigues' formula

$$(1.7.18)\quad L_n^{(\alpha)}(x,r,p) = \frac{x^{-\alpha}\, e^{px^r}}{n!}\, \frac{d^n}{dx^n}\, (x^{\alpha+n}\, e^{-px^r}) \ .$$

Chatterjea [12] and Kharadze [34] gave the generalizations of Hermite polynomials which are based on $D_k^k$ operator.

For particular interest Chatterjea [9] has studied the generalized Bessel polynomials defined by

$$(1.7.19)\quad M_n^k(x,a,b) = b^{-n}\, x^{k-a-(k-2)n}\, e^{b/x}\, D^n(x^{kn+a-k}\, e^{-b/x}),$$

which is, undoubtedly, the particular case of (1.7.17). Further Chatterjea [11] has defined a generalized function by the Rodrigues' formula

$$(1.7.20)\quad F_r^{(n)}(x;a,b,k,p) = x^{-a}\, e^{px^r}\, D^n(x^{kn+a}\, e^{-px^r}).$$

It includes, as special cases, the Hermite, Laguerre, Bessel polynomials and the generalized Hermite function of Gould and Hopper.

Chandel [19] introduced a class of polynomials defined by Rodrigues' formula

$$(1.7.21)\quad T_n^{\alpha,k}(x,r,p) = x^{-\alpha}\, e^{px^r}\, \Omega_x^n\, \{x^{\alpha}\, e^{-px^r}\},$$

where

$$(1.7.22)\quad \Omega_x = x^k \frac{d}{dx} \ .$$

Further Chandel [20] gave its generalization in the form

$$(1.7.23) \qquad G_n(h,g,k) = e^{-hg(x)} \; \Omega_x^n \, e^{hg(x)},$$

where h,k ($\neq 1$) are constants and g is any function of x.

Recently Srivastava and Singhal [56] gave the extension of (1.7.23), in the following way :

$$(1.7.24) \qquad T_n^{(\alpha,\beta)}(x;a,b,c,d,p,r)$$

$$= \frac{(ax+b)^{-\alpha} \, (cx+d)^{-\beta} \, e^{px^r}}{n!} \, D_x^n[(ax+b)^{\alpha+n}(cx+d)^{\beta+n}e^{-px^r}] \,,$$

$$D_x \equiv \frac{d}{dx} \,.$$

Further Srivastava and Panda [57] made a systematic study of this class by introducing the polynomials defined by

$$(1.7.25) \qquad S_n^{(\alpha,\beta)} [\, x,a,b,c,d;\gamma,\varepsilon;\; w(x) \,]$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}}{n\,!\; \omega(x)} \, D_x^n\{(ax+b)^{\gamma n+\alpha}(cx+d)^{n\varepsilon+\beta} \; w(x)\}$$

where $a,b,c,d,\alpha,\beta,\gamma,\varepsilon$ are arbitrary constants and $w(x)$ is independent of n and differentiable an arbitrary number of times.

In order to present various interesting operational relationships Chandel and Agrawal [22] have considered

$$(1.7.26) \quad T_{n,e,f,g}^{(\alpha,\beta,\gamma,k)}(x;a,b,c,d,p,r)$$

$$= \frac{(ax+b)^{-\alpha}(cx+d)^{-\beta}x^{-\gamma}e^{px^r}}{n!}\,\Omega_x^n\,[(ax+b)^{\alpha+en}(cx+d)^{\beta+fn}x^{\gamma+gn}e^{-px^r}].$$

where $a,b,c,d,e,f,g,p,r,\alpha,\beta,\gamma,k$ $(\neq 1)$ are arbitrary constant independent of n.

Recently Chandel and Bhargava [21] studied a generalization of several polynomial systems defined by

$$(1.7.27) \quad G_n(a,k;h,g(x)) = e^{-hg(x)}\,T_{a,k}^n\,(e^{hg(x)})$$

where h,a,k are constants, g(x) is differentiable function of x and

$$T_{a,k} = x^k(a+xD).$$

Very recently, Bhargava [6] further gave the generalization of (1.7.27) to introduce a new sequence of functions defined by

$$(1.7.28) \quad G_n\,[a,k,p;g(x),h(x)] = e^{-pg(x)}T_{a,k}^n\{\,[h(x)]^n\,e^{pg(x)}\}$$

where h(x), g(x) are suitable functions of x, and a,k,p are independent of x.

(c) <u>Generalization of differential equation</u>. The differential equation for associated Legendre functions is given by

$$(1.7.29) \quad (1-z^2)\,\frac{d^2\omega}{dz^2} - 2z\,\frac{d\omega}{dz} + \{n(n+1) - \frac{m^2}{1-z^2}\}\,\omega = 0$$

Kuipers and Meulenbeld [36] have given a generalization of the above equation by

$$(1.7.30)\quad (1-z^2)\frac{d^2\omega}{dz^2} - 2z\frac{d\omega}{dz} + \{n(n+1) - \frac{n^2}{2(1+z)} - \frac{m^2}{2(1-z)}\}\,\omega = 0.$$

The solutions of the above differential equation are $P_k^{m,n}(z)$ and $Q_k^{m,n}(z)$.

The complete solution is given by

$$\omega = A\, P_k^{m,n}(z) + B\, Q_k^{m,n}(z)$$

Kuipers and Meulenbeld have made a detailed study of these functions in the series of papers.

## 1.8 Hypergeometric functions of one variable.

In the study of second order linear differential equations with three regular singular points, there arises the function

$$(1.8.1)\quad {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\;\; n!}\, z^n,$$

for c neither zero nor negative integer. Euler obtained many properties of the ${}_2F_1$ but a detailed and systematic study was made by Gauss. Therefore this function is also called Gauss' function. The above hypergeometric function has been generalized in different ways by Hein, Appell and others. The generalized hypergeometric function is defined by

$$(1.8.2)\quad {}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\ldots(a_p)_n}{(b_1)_n(b_2)_n\ldots(b_q)_n}\,\frac{z^n}{n!},$$

where

$$(a)_n = a(a+1) \ldots (a+n-1),\ n \geq 1$$

$$(a)_o = 1,\ a \neq 0$$

This function is called generalized Gauss' function or generalized hypergeometric function. The name 'hypergeometric' (from the Greek $\nu^c \pi\ \varepsilon\ \rho$, above or <u>beyond</u>) was first used by J. Wallis in his work <u>Arithmetica Infiniterum</u> (1655) for the series whose nth term was $a(a+b)\ldots[a+(n-1)b]$.

Euler used the term 'hypergeometric' in this <u>sense</u>, the modern use of the term being apparently due to Kummer (Journal für Math. XV (1836)).

It is being assumed that no denominator parameter $b_j$ is zero or a negative integer because in that case the series is not defined. The series ${}_pF_q$ is convergent for all finite z, real or complex, if $p \leq q$ and for $|z| < 1$, if $p = q+1$.

For $p = q+1$, the series is absolutely convergent on the circle $|z| = 1$, if $\mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right) > 0$.

If any numerator parameter $a_j$ in (1.8.2) is negative integer, the series terminates and the function reduces to a polynomial.

The hypergeometric series was generalized in a different way by E. Heine (1878), who considered the series

$$(1.8.3)\quad 1 + \frac{(1-q^a)(1-q^b)z}{(1-q)^c(1-q)} + \frac{(1-q^a)(1-q^{a+1})(1-q^b)(1-q^{b+1})\ z^2}{(1-q)^c(1-q)^{c+1}(1-q)(1-q^2)} + \ldots,$$

where $|q| < 1$, so that as $q \to 1$, this serves $\to$ Gauss' series ${}_2F_1\,(a,b;c;z)$ and is known as the q-analogue of Gauss' hypergeometric function.

More generally,

$$(1.8.4)\quad {}_A\Phi_B\left[\begin{matrix} a_1,\ldots,a_A; \\ b_1,\ldots,b_B; \end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{[a_1]_n\,[a_2]_n\,\cdots[a_A]_n\quad z^n}{[b_1]_n\,[b_2]_n\,\cdots[b_B]_n\,[q]_n},$$

where

$$[a]_n = (1-a)(1-aq)\ldots(1-aq^{n-1});\; n = 1,2,3,\ldots$$

$$[a]_o = 1$$

The above series is called generalized basic hypergeometric series, which converges for $|z| < 1$, when $A \leq B+1$ and $|q| < 1$.

1.9 Hypergeometric series of two variables. The great success of the theory of hypergeometric series in one variable has stimulated the development of a corresponding theory in two or more variables. In (1880), Appell has defined four series $F_1$, $F_2$, $F_3$ and $F_4$, which are analogous to Gauss' ${}_2F_1(a,b;c;z)$.

Horn puts

$$f(m,n) = \frac{F(m,n)}{F'(m,n)}\,,\; g(m,n) = \frac{G(m,n)}{G'(m,n)}\,,$$

where $F$, $F'$, $G$, $G'$ are polynomials in $m,n$ of respective degrees $p,p',q,q'$. $F'$ is assumed to have a factor $(m+1)$, and $G'$ a factor $(n+1)$; $F$ and $F'$ have no common factor except possibly,

m+1; and G and G′ have no common factor except possibly n+1. The greatest of the four numbers p,p′,q,q′, is the order of the hypergeometric series. Horn investigated, in particular, the hypergeometric series of order two and found that, apart from certain series which are either expressible in terms of one variable or are products of two hypergeometric series, each in one variable, there are essentially thirty four distinct convergent series of order two (Horn 1931, corrections in Borngässer 1933). There are fourteen complete series for which $p = p' = q = q' = 2$, which are given in pioneer work of Erdélyi [23]. There are twenty confluent series which are limiting forms of the complete ones and for which $p < p' = 2$, $q \leq q' = 2$ and p,q not both equal to 2.

Srivastava and Daoust [53] defined a generalization of the Kampé de Fériet function (1926) by means of the double hypergeometric series

$$(1.9.1)\quad S^{A:B;B'}_{C:D;D'}\left[\begin{matrix}[(a):\theta,\Phi]:[(b):\Psi];\ [(b'):\Psi'];\\ [(c):\delta,\varepsilon]:(d):\eta];\ [(d'):\eta'];\end{matrix}\ x,y\right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{A}\Gamma(a_j+m\theta_j+n\Phi_j)\prod_{j=1}^{B}\Gamma(b_j+m\,\Psi_j)\prod_{j=1}^{B'}\Gamma(b'_j+n\,\Psi'_j)}{\prod_{j=1}^{C}\Gamma(c_j+m\delta_j+n\varepsilon_j)\prod_{j=1}^{D}\Gamma(d_j+m\eta_j)\prod_{j=1}^{D'}\Gamma(d'_j+n\eta'_j)}\ \frac{x^m\, y^n}{m!\,n!},$$

where the coefficients

$$\left\{\begin{matrix}\theta_1,\ldots,\theta_A;\ \Phi_1,\ldots,\Phi_A;\ \Psi_1,\ldots,\Psi_B;\ \Psi'_1,\ldots,\Psi'_{B'};\\ \delta_1,\ldots,\delta_C;\ \varepsilon_1,\ldots,\varepsilon_C;\ \eta_1,\ldots,\eta_D;\ \eta'_1,\ldots,\eta'_{D'}\end{matrix}\right.$$

are real and positive; and for brevity (a) is taken to denote the sequence of A parameters $a_1, \ldots, a_A$; with similar interpretations for (b), (b′), etc.

1.10. Hypergeometric functions in three variables. In the year 1893 Lauricella [37] defined the four hypergeometric functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ of three variables and also conjectured the existence of ten more hypergeometric functions of three variables. Later on these ten functions were defined by Saran [45]. Further Srivastava [51] also introduced three new hypergeometric functions of three variables viz. $H_A$, $H_B$, $H_C$, which were not covered by Lauricella's conjecture nor was their existence noticed by earlier writer~~s~~

1.11. Hypergeometric functions of four variables. Until Exton [25,pp. 77-79] defined and examined a few of their properties, no specific study had been made of any hypergeometric functions of four variables, apart from the four Lauricella functions $F_A^{(4)}$, $F_B^{(4)}$, $F_C^{(4)}$ and $F_D^{(4)}$ and certain of their limiting cases. On account of the large number of such functions which arise from a systematic study of all the possibilities. He restricted himself to those functions which are complete and of the second order and which involve at least one product of the type (a,k+m+n+p) in series representation; k,m,n,p are the indices of quadruple summation. Here we write only those series which will be

used in our investigations.

(1.11.1) $K_2(a,a,a,a;b,b,b,c;d_1,d_2,d_3,d_4;x,y,z,t)$

$$= \Sigma \frac{(a,k+m+n+p)(b,k+m+n)(c,p)x^h y^m z^n t^p}{(d_1,k)(d_2,m)(d_3,n)(d_4,p)\ k!\,m!\,n!\,p!},$$

(1.11.2) $K_5(a,a,a,a;b_1,b_1,b_2,b_2;c_1,c_2,c_3,c_4;x,y,z,t)$

$$= \Sigma \frac{(a,k+m+n+p)(b_1,k+m)(b_2,n+p)\ x^k\ y^m\ z^n\ t^p}{(c_1,k)(c_2,m)(c_3,n)(c_4,p)\ k!\ m!\ n!\ p!}$$

(1.11.3) $K_{11}(a,a,a,a;b_1,b_2,b_3,b_4;c,c,c,d;x,y,z,t)$

$$= \Sigma \frac{(a,k+m+n+p)(b_1,k)(b_2,m)(b_3,n)(b_4,p)\ x^k y^m z^n t^p}{(c,k+m+n)\quad (d,p) \qquad\qquad k!\,m!\,n!\,p!}$$

(1.11.4) $K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;x,y,z,t)$

$$= \Sigma \frac{(a,k+m+n+p)(b_1,k)(b_2,m)(b_3,n)(b_4,p)x^k y^m z^n t^p}{(c_1,k+m)\quad (c_2,n+p) \qquad\qquad k!\,m!\,n!\,p!}$$

(1.11.5) $K_{15}(a,a,a,b_5;b_1,b_2,b_3,b_4;c,c,c,c;x,y,z,t)$

$$= \Sigma \frac{(a,k+m+n)(b_5,p)(b_1,k)(b_2,m)(b_3,n)(b_4,p)x^k y^m z^n t^p}{(c,k+m+n+p) \qquad\qquad k!\ m!\ n!\ p!}$$

(1.11.6) $K_{20}(a_1,a_1,b_3,b_4;b_1,b_2,a_2,a_2;c,c,c,c;x,y,z,t)$

$$= \Sigma \frac{(a_1,k+m)(b_3,n)(b_4,p)(b_1,k)(b_2,m)(a_2,n+p)x^k y^m z^n t^p}{(c,k+m+n+p) \qquad\qquad k!\ m!\ n!\ p!}$$

1.12. <u>Multiple hypergeometric functions of several variables</u>. While several authors, for example, Green (1834), Hermite (1865) and Didon (1870) have discussed what amount to certain specialised multiple hypergeometric functions, it was left to Lauricella [37] to approach this topic systematically. Beginning with the Appell functions Lauricella proceeded to define and study the four important functions which bear his name. They have the following multiple series representations :

(1.12.1) $$F_A^{(n)}(a,b_1,\ldots,b_n;c_1,\ldots,c_n;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c_1,m_1)\ldots(c_n,m_n)} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \quad |x_1|+\ldots+|x_n| < 1,$$

(1.12.2) $$F_B^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a_1,m_1)\ldots(a_n,m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_n)} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \quad |x_1| < 1,\ldots,|x_n| <1,$$

(1.12.3) $$F_C^{(n)}(a,b;c_1,\ldots,c_n;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b,m_1+\ldots+m_n)}{(c_1,m_1)\ldots(c_n,m_n)} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}, \quad |x_1^{\frac{1}{2}}|+\ldots+|x_n^{\frac{1}{2}}| < 1,$$

$$(1.12.4)\quad F_D^{(n)}(a,b_1,\ldots,b_n;c;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_n)} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}, \ |x_1| < 1,\ldots,|x_n| < 1.$$

A number of following confluent forms of the Lauricella functions $F_A^{(n)}$, $F_B^{(n)}$, $F_C^{(n)}$ and $F_D^{(n)}$ exist and which may be obtained by limiting processes :

$$(1.12.5)\quad \Psi_2^{(n)}(a;c_1,\ldots,c_n;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)}{(c_1,m_1)\ldots(c_n,m_n)} \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!},$$

$$(1.12.6)\quad \Phi_2^{(n)}(b_1,\ldots,b_n;c;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}$$

$$(1.12.7)\quad \Phi_D^{(n)}(a,b_1,\ldots,b_{n-1},-;c;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_n)(b_1,m_1)\ldots(b_{n-1},m_{n-1})}{(c,m_1+\ldots+m_n)} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}$$

These three functions have been mentioned previously by Humbert (1920), Erdélyi (1937) and Srivastava and Exton (1973) respectively and generalize the Humbert functions $\Psi_2$, $\Phi_2$ and $\Phi_1$. Several other limiting forms of the Lauricella functions may be obtained such as

(1.12.8) $\Xi_1^{(n)}(a_1,\ldots,a_n,b_1,\ldots,b_{n-1};c;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a_1,m_1)\ldots(a_n,m_n)(b_1,m_1)\ldots(b_{n-1},m_{n-1})}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!}$$

(1.12.9) $\Phi_3^{(n)}(b_1,\ldots,b_{n-1};c;x_1,\ldots,x_n)$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(b_1,m_1)\ldots(b_{n-1},m_{n-1})}{(c,m_1+\ldots+m_n)} \frac{x_1^{m_1}}{m_1!}\ldots\frac{x_n^{m_n}}{m_n!},$$

which generalize the functions $\Xi_1$ and $\Phi_3$ respectively. These are the most interesting of the multiple confluent hypergeometric functions on account of their applications.

Recently Exton [24] considered the two multiple hypergeometric functions closely related to Lauricella function $F_D^{(n)}$ and which follow as generalizations of certain of the quadruple functions discussed in previous section 1.11. These functions are defined as

$$(1.12.10)\quad {}^{(k)}_{(1)}E_D^{(n)}\,(a,b_1,\ldots,b_n;c,c';x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1,\ldots,m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_k)(c',m_{k+1}+\ldots+m_n)} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!},$$

$$(1.12.11)\quad {}^{(k)}_{(2)}E_D^{(n)}(a,a',b_1,\ldots,b_n;c;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(a',m_{k+1}+\ldots+m_n)(b_1,m_1)\ldots(b_n,m_n)}{(c,m_1+\ldots+m_n)}$$

$$\times \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}$$

Prompted by this work Chandel [17] defined and studied the following function closely related to Lauricella $F_C^{(n)}$ :

$$(1.12.12)\quad {}^{(k)}_{(1)}F_C^{(n)}(a,a',b;c_1,\ldots,c_n;x_1,\ldots,x_n)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a,m_1+\ldots+m_k)(a'm_{k+1}+\ldots+m_n)(b,m_1+\ldots+m_n)}{(c_1,m_1),\ldots,(c_n,m_n)} \times$$

$$\times \frac{x_1^{m_1}}{m_1!} \ldots \frac{x_n^{m_n}}{m_n!}.$$

If $|x_i| < r_i$

where $r_i$, $i = 1,\ldots,n$ are the associated radii of convergence of the series then

(a) $\,^{(k)}_{(1)}E^{(n)}_{D}$ is convergent if $r_k + r_n = 1$, with $r_1 = \dots = r_k$,

$r_{k+1} = \dots = r_n$.

(b) $\,^{(k)}_{(2)}E^{(n)}_{D}$ is convergent if $r_k + r_n = r_k \cdot r_n$, with $r_1 = \dots = r_k$,

$r_{k+1} = \dots = r_n$,

and (c) $\,^{(k)}_{(1)}E^{(n)}_{C}$ is convergent if $(\sqrt{r_1} + \dots + \sqrt{r_k})^2 + (\sqrt{r_{k+1}} + \dots + \sqrt{r_n})^2 = 1$

Srivastava and Daoust [54] defined a most generalized multiple hypergeometric series by

$$(1.12.13) \qquad S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$= S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}} \left[ \begin{matrix} [(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \; x_1,\dots,x_n \right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{\prod_{j=1}^{A} \Gamma(a_j + \sum_{i=1}^{n} m_i \theta_j^{(i)}) \prod_{j=1}^{B'} \Gamma(b_j' + m_1 \phi_j') \cdots \prod_{j=1}^{B^{(n)}} \Gamma(b_j^{(n)} + m_n \phi_j^{(n)})}{\prod_{j=1}^{C} \Gamma(c_j + \sum_{i=1}^{n} m_i \psi_j^{(i)}) \prod_{j=1}^{D'} \Gamma(d_j' + m_1 \delta_j') \cdots \prod_{j=1}^{D^{(n)}} \Gamma(d_j^{(n)} + m_n \delta_j^{(n)})} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

or alternatively by

$$(1.12.14)\quad F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\Phi'];\dots;[(b^{(n)}):\Phi^{(n)}];\\ [(c):\Psi',\dots,\Psi^{(n)}]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\dots,x_n\right]$$

$$= \sum_{m_1,\dots,m_n=0}^{\infty} \frac{\prod_{j=1}^{A}(a_j)_{m\theta_j'+\dots+m_n\theta_j^{(n)}} \prod_{j=1}^{B'}(b_j)_{m_1\Phi_j'}\cdots\prod_{j=1}^{B^{(n)}}(b_j^{(n)})_{m_n\Phi_j^{(n)}}}{\prod_{j=1}^{C}(c_j)_{m_1\Psi'+\dots+m_n\Psi_j^{(n)}}\prod_{j=1}^{D'}(d_j')_{m_1\delta_j'}\cdots\prod_{j=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_j^{(n)}}}\,\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

where

$$(1.12.15)\quad \theta_j^{(i)},\ j=1,\dots,A;\ \Phi_j^{(i)}, j=1,\dots,B^{(i)};\ \Psi_j^{(i)}, j=1,\dots,C;$$

$$\delta_j^{(i)},\ j=1,\dots,D^{(i)};\ 1\le i\le n;$$

are real and positive and (a) is taken to abbreviate the sequence of A parameters $a_1,\dots,a_A$; $(b^{(i)})$ abbreviates the sequence of $B^{(i)}$ parameters $b_1^{(i)},\dots,b_{B^{(i)}}^{(i)}$, $i=1,\dots,n$; with similar interpretations for (c) and $(d^{(i)})$, $i=1,\dots,n$; etc.

These multiple hypergeometric functions will be frequently used in our investigations.

1.13. Fox H-function. Fox $p > q+1$, originally the G-function was defined by Meijer (1936) and E-function by Mac Robert (1937-38) by hypergeometric series ${}_pF_q$. E-function is itself the particular case of G-function. Later on in 1946 Meiger [41] replaced the definition of G-function by one in terms of Mellin-Barnes type integral. Fox [27] in 1961 introduced a most generalized function H of one variable, which includes Meiger's G and Mac Robert E-functions as special cases.

The H-function is defined as

$$(1.13.1)\quad H(x) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\prod_{j=1}^{q}\Gamma(b_j+c_js)\prod_{j=1}^{p}\Gamma(a_j-e_js)\,x^{-s}\,ds}{\prod_{j=1}^{q}\Gamma(b_j+c_j-c_j\ s)\prod_{j=1}^{p}\Gamma(a_j-e_j+c_js)},$$

where $c_j > 0$, $j = 1,\ldots,q$; $e_j > 0$, $j = 1,\ldots,p$; $q \geq p+1$ and the poles of the integrand in (1.13.1) are simple.

Gupta [31] defined this function with slight difference in parameters, in the form

$$(1.13.2)\quad H^{m,n}_{p,q}\left[x\left|\begin{matrix}(a_1,e_1),(a_2,e_2),\ldots,(a_p,e_p)\\(b_1,f_1),(b_2,f_2),\ldots,(b_q,f_q)\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_js)\prod_{j=1}^{n}\Gamma(1-a_j+e_js)\,x^s\,ds}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_js)\prod_{j=n+1}^{p}\Gamma(a_j-e_j\ s)},$$

where L is a suitable contour of Barnes type such that the poles of $\Gamma(b_j-f_js)$, $j = 1,\ldots,m$, lie on the right and those of $\Gamma(1-a_j+e_js)$, $j = 1,\ldots,n$, on the left of the contour. An empty product is to be interpreted as 1, $0 \leq m \leq q$; $0 \leq n \leq p$, e's and f's are all positive. Also the parameters are so restricted that the integral on the right of (1.13.2) is convergent.

We remark in passing that a study of one form or the other of the H-function, which was initiated as long ago as 1888 by S. Pincherle, appeared in the works of E.W. Barnes in 1908, H. Mellin in 1910, A.L. Dixon and W.L. Ferrar in 1936, S. Bochner in 1958 and several others (cf., e.g., [23], p. 49, §1.19 for details). A first systematic discussion of the properties of the H-function as a symmetrical Fourier Kernel was incorporated by C. Fox. [27, p. 408], whose name seems to have been associated with this function in the literature ever since.

The Meijer's G-function was generalized by Agrawal [3] in two variables. Several definitions and notations of the H-function of two complex variables have appeared in the literature. We find it convenient to employ the following contracted notation for the double H-function of Mittal and Gupta [42] defined by

$$(1.13.3)\quad H^{o,h:(m,n);(r,s)}_{k,\ell:[p,q];[u,v]}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}((\mu_k;E_k,E'_k)):((\alpha_p,A_p));((\gamma_u,C_u))\\((\nu_\ell;F_\ell,F'_\ell)):((\beta_q,B_q));((\delta_v,D_v))\end{matrix}\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\theta(\xi)\,\Phi(\eta)\,\Psi(\xi,\eta)\,x^{\xi}y^{\eta}\,d\xi\,d\eta,$$

with

$$(1.13.4)\quad \theta(\xi) = \frac{\prod_{j=1}^{m}\Gamma(\beta_j-B_j\xi)\prod_{j=1}^{n}\Gamma(1-\alpha_j+A_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j+B_j\xi)\prod_{j=n+1}^{p}\Gamma(\alpha_j-A_j\xi)},$$

$$(1.13.5)\quad \Phi(\eta) = \frac{\prod_{j=1}^{r}\Gamma(\delta_j-D_j\eta)\prod_{j=1}^{s}\Gamma(1-\gamma_j+C_j\eta)}{\prod_{j=r+1}^{p}\Gamma(1-\delta_j+D_j\eta)\prod_{j=s+1}^{u}\Gamma(\gamma_j-C_j\eta)},$$

and

$$(1.13.6)\quad \Psi(\xi,\eta) = \frac{\prod_{j=1}^{h}\Gamma(1-\mu_j+E_j\xi+E'_j\eta)}{\prod_{j=h+1}^{k}\Gamma(\mu_j-E_j\xi-E'_j\eta)\prod_{j=1}^{\ell}\Gamma(1-\nu_j+F_j\xi+F'_j\eta)},$$

where an empty product is interpreted as 1, the integers $h,k,\ell,m,n,p,q,r,s,u,v$ are such that $0 \le h \le k$, $\ell \ge 0$, $0 \le m \le q$, $0 \le n \le p$, $0 \le r \le v$, $0 \le s \le u$, the coefficients $E_j$, $E'_j$, $F_j$, $F'_j$, as also $A_j$, $B_j$, $C_j$, $D_j$, are all positive, and the sequences of parameters $(\alpha_p)$, $(\beta_q)$, $(\gamma_u)$, $(\delta_v)$, $(\mu_k)$ and $(\nu_\ell)$ are so restricted that none of the poles of the integrand coincide. The contour $L_1$ in the complex $\xi$-plane and contour $L_2$ in the complex $\eta$-plane, are of the Mellin-Barnes type with indentations, if necessary, to ensure that they separate one set

of poles from the order.

Recently, using the abbreviation (a) to denote the set of A parameters $a_1,\ldots,a_A$, and $(b^{(i)})$ to denote the set of $B^{(i)}$ parameters $b_1^{(i)},\ldots,b_{B^{(i)}}^{(i)}$, $i = 1,\ldots,n$ with similar interpretations for (c), $(d^{(i)})$, etc., Srivastava and Panda [55] defined on extension of the H-function in several complex variables $x_1,\ldots,x_n$ by means of the multiple integral

$$(1.13.7)\quad H^{o,\lambda:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\Psi',\ldots,\Psi^{(n)}]:[(d'):(\delta')];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix} x_1,\ldots,x_n\right]$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\ldots\int_{L_n}\Phi_1(\xi_1)\ldots\Phi_n(\xi_n)\quad(\xi_1,\ldots,\xi_n)x_1^{\xi_1}\ldots x_n^{\xi_n}\,d\xi_n,$$

where $\omega = \sqrt{-1}$,

$$(1.13.8)\quad \Phi_i(\xi_i) = \frac{\prod_{j=1}^{\mu^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\xi_i]\prod_{j=1}^{\nu^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\xi_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}\xi_j]\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}\xi_i)},$$

$i = 1,\ldots,n;$

$$(1.13.9)\quad \Psi(\xi_1,\ldots,\xi_n) = \frac{\prod_{j=1}^{\lambda}\Gamma[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\xi_i]}{\prod_{j=\lambda+1}^{A}\Gamma[a_j-\sum_{j=1}^{n}\theta_j^{(i)}\xi_i]\prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{n}\Psi_j^{(i)}\xi_i]},$$

an empty product is to be interpreted as 1, the coefficients $\theta_j^{(i)}$, $j = 1,\ldots,A$; $\phi_j^{(i)}$, $j = 1,\ldots,B^{(i)}$, $\Psi_j^{(i)}$, $j = 1,\ldots,C$; $\delta_j^{(i)}$, $j = 1,\ldots,D^{(i)}$, and $i = 1,\ldots,n$, are positive numbers, and $\lambda$, $\mu^{(i)}$, $\nu^{(i)}$, A, $B^{(i)}$, C, $D^{(i)}$ are integers such that $0 \le \lambda \le A$, $0 \le \mu^{(i)} \le D^{(i)}$, $C \ge 0$, and $0 \le \nu^{(i)} \le B^{(i)}$, $i = 1,\ldots,n$. The contour $L_i$ in the complex $\xi_i$ plane is of the Mellin-Barnes type and, as in the cases of one and two variables, it separates one set of poles of the integrand from the order, the various parameters being so restricted that none of these poles coincide. If we let

(1.13.10) $$\Delta_i = \sum_{j=1}^{\lambda} \theta_j^{(i)} - \sum_{j=\lambda+1}^{A} \theta_j^{(i)} + \sum_{j=1}^{\nu^{(i)}} \Phi_j^{(i)} - \sum_{j=\nu^{(i)}+1}^{B^{(i)}} \Phi_j^{(i)}$$

$$- \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{\mu^{(i)}} \delta_j^{(i)} - \sum_{j=\mu^{(i)}+1}^{D^{(i)}} \delta_j^{(i)}, \quad i = 1,\ldots,n,$$

then it can be easily seen that for $\Delta_i > 0$, $i = 1,\ldots,n$ and with the points $x_i = 0$, $i = 1,\ldots,n$, being tacitly excluded, the multiple contour integral will converge absolutely and define an H-function of n variables, analytic in the sectors given by

(1.13.11) $$|\arg (x_i)| < \tfrac{1}{2} \Delta_i \pi, \quad i = 1,\ldots,n.$$

The above function is most generalized function of several variables.

1.14. <u>Brief survey of the chapters</u>. In chapter II, we unify the study of two general classes of the polynomials studied by Gould [30] and Srivastava [52]. Further we extend this work by introducing some polynomial systems of several variables by means of their generating function. In the second part of this chapter, we further unify the study of three general classes of polynomials considered by Gould [30], Srivastava [52] and Panda [43] by introducing another polynomial system of several variables. Some interesting special cases of these polynomials have also been discussed.

Chapter III deals with the applications of Srivastava theorem (1979) to extend the work of Srivastava and Buschman (1975). A binomial analog of Srivastava theorem has also been introduced and its applications have been shown to extend the work of Srivastava and Buschman (1975) and the works of several others also.

Chapter IV gives some applications of the operator $\Omega_x = \lambda x^k + x^{k+1} \frac{\partial}{\partial x}$ to establish the operational representations of certain multiple hypergeometric functions of several variables. Some of their special cases have also been discussed to obtain operational representations of hypergeometric functions of four variables introduced by Exton [25].

Chapter V introduces a multidimensional Whittaker transform to evaluate certain new multiple integrals involving multiple hypergeometric functions of several variables.

Chapter VI deals with the introduction of another multidimensional Whittaker transform to establish new integral representations of certain generalized hypergeometric functions of several variables.

Chapter VII introduces two multidimensional Gauss' transforms to evaluate certain new integral representations of certain generalized hypergeometric functions of several variables.

In Chapter VIII on applications first we employ generalized multiple hypergeometric function of Srivastava and Daoust [54] to obtain the formal solution of fundamental differential equation of cooling of an infinitely long cylinder of given radius, heated to the temperature $u_o = f(z)$ (z is the distance from the axis) and radiating heat into the surrounding medium at zero temperature.

In second part of this chapter we evaluate certain integrals involving multiple hypergeometric functions of Srivastava and Daoust [54] and their applications will be shown in solving a problem on heat conduction given by Bhonsle (1966) and in establishing some expansion formulae involving the above function.

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CHAPTER II

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# GENERATING FUNCTIONS FOR CERTAIN POLYNOMIAL SYSTEMS OF SEVERAL VARIABLES

2.1 Introduction. Gould [4] considered a class of generalized Humbert polynomials defined by

$$(2.1.1) \qquad (C-mxt+yt^m)^p = \sum_{n=0}^{\infty} P_n(m,x,y,p,C)\ t^n,$$

where m is a positive integer and other parameters are unrestricted in general.

Recently, Srivastava [6] considered the class of generalized Hermite polynomials defined by the generating function

$$(2.1.2) \qquad \sum_{n=0}^{\infty} \gamma_n^m(x)\ \frac{t^n}{n!} = G(mxt-t^m),$$

where

$$(2.1.3) \qquad G(z) = \sum_{n=0}^{\infty} c_n\ z^n, \qquad c_o \neq 0$$

and m is an arbitrary positive integer.

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Following two papers of this chapter have been published :

1. A note on a generating functions for certain polynomial systems, Ranchi Univ. Math.J. 10 (1979), 61-66.
2. Some generating functions for certain polynomial systems of several variables, Proc. Nat. Acad. Sci. India, Sect.A,51 II (1981),133-138.

In the first part of this chapter we unify the study of the polynomial systems generated by (2.1.1) and (2.1.2) by means of the generating function

$$(2.1.4) \qquad G(C-mxt+yt^q) = \sum_{n=0}^{\infty} g_n(m,x,y,q,C)\, t^n,$$

where $G(z)$ is any function of $z$, $q$ is an arbitrary positive integer, and other parameters are unrestricted in general,

For $G(z) = \sum_{n=0}^{\infty} c_n z^n$, $c_o \neq 0$, $C = 0$, $q = m$, $y = -1$ and on replacing $m$ by $-m$ in (2.1.4), we get

$$(2.1.5) \qquad \gamma_n^m(x) = n!\, g_n(-m,x,-1,m,0).$$

On the other hand, if we set

$$(2.1.6) \qquad G(z) = z^p,$$

we get

$$(2.1.7) \qquad P_n(q,x,y,p,C) = g_n(m, \tfrac{qx}{m}, y,q,C).$$

For brevity, we shall write $g_n$ for $g_n(m,x,y,q,c)$ throughout this chapter.

2.2 Some Theorems. We state our first set of results given by

Theorem 1. For every sequence $\{g_n\}$, defined by (2.1.4),

$$(2.2.1) \qquad D_x\, g_o = 0$$

(2.2.2) $$mng_n = mxD_x\, g_n - qyD_x\, g_{n-q+1},\quad n \geq q-1$$

and

(2.2.3) $$n\, g_n = x\, D_x\, g_n,\quad 1 \leq n \leq q-2$$

where

$$D_x \equiv \frac{\partial}{\partial x}\ .$$

Proof. If we let

(2.2.4) $$F = G(C-mxt+yt^q),$$

then

(2.2.5) $$\frac{\partial F}{\partial x} = -mt\ G', \frac{\partial F}{\partial t} = (-mx+qyt^{q-1})\ G'.$$

Therefore

(2.2.6) $$mt\ \frac{\partial F}{\partial t} = (mx-qy\ t^{q-1})\ \frac{\partial F}{\partial x}$$

Since

$$F = \sum_{n=0}^{\infty} g_n\ t^n,$$

therefore, equating the coefficients of $t^n$ on both the sides, theorem 1 follows.

Similarly, we may prove

Theorem 2. For every sequence $\{g_n\}$, defined by (2.1.4),

(2.2.7) $$q\, y\, D_y\, g_{n-q+1} = m\, x\, D_y\, g_n + (n-q+1)g_{n-q+1},\quad n \geq q-1$$

and

(2.2.8) $$D_y\, g_n = 0,\ 0 \leq n \leq q-2.$$

2.3. Applications of theorem 1. Replacing m by $-m$ and taking $C = 0$, $q = m$, $y = -1$ in theorem 1, we obtain

Corollary 2.1.1. For every sequence $\{\gamma_n^m\}$, defined by (2.1.2),

(2.3.1) $$D_x \gamma_0^m = 0 ,$$

(2.3.2) $$n \gamma_n^m = x D_x \gamma_n^m + (-1)^m (-n)_{m-1} D_x \gamma_{n-m+1}^m, \quad n \geq m-1$$

and

(2.3.3) $$n \gamma_n^m = x D_x \gamma_n^m , \quad 1 \leq n \leq (m-2).$$

Again, taking $q = m$, we get

Corollary 2.1.2. For every sequence $\{P_n\}$, defined by (2.1.1)

(2.3.4) $$D_x P_0 = 0,$$

(2.3.5) $$n P_n = x D_x P_n - y D_x P_{n-m+1}, \quad n \geq m-1,$$

and

(2.3.6) $$n P_n = x D_x P_n, \quad 1 \leq n \leq m-2,$$

where

$$P_n = P_n(m,x,y,p,C).$$

2.4. Applications of theorem 2. For $q = m$, theorem 2 yields

Corollary 2.2.1. For every sequence $\{P_n\}$ defined by (2.1.1),

(2.4.1) $\quad m\, y\, D_y\, P_{n-m+1} = mxD_y\, P_n + (n-m+1)\, P_{n-m+1}, \quad n \geq m-1$

and

(2.4.2) $\quad D_y\, P_n = 0, \quad 0 \leq n \leq m-2$

2.5. Miscellaneous results. If

(2.5.1) $$G(z) = \sum_{n=0}^{\infty} c_n\, z^n,$$

then we easily have

(2.5.2) $$g_n = \sum_{i=0}^{\infty} c_i\, P_n\left(q, \frac{mx}{q}, y, i, c\right),$$

and

(2.5.3) $$\gamma_n^m(x) = n! \sum_{i=0}^{[n/m]} c_{n-(m-1)i} \binom{n-(m-1)i}{i} (-1)^i\, (mx)^{n-mi} .$$

Starting with (2.2.4), we also establish

(2.5.4) $$D_x\, g_{n-q+1} + m\, D_y\, g_n = 0, \quad n \geq q-1.$$

Now an appeal to (2.2.7) and (2.5.4) shows that

(2.5.5) $$(qyD_y + x\, D_x - n)\, g_n = 0.$$

For $q = m$, (2.5.4) and (2.5.5) give

(2.5.6) $$D_x\, P_{n-m+1} + m\, D_y\, P_n = 0, \quad n \geq m-1$$

and

(2.5.7) $$(myD_y + x\, D_x - n)\, P_n = 0, \text{ respectively.}$$

2.6 Extension of (2.1.4). Here in this section we further generalize (2.1.4) and introduce some polynomial systems of several variables by means of the generating function

$$(2.6.1)\quad G(a_o+a_1x_1t+\dots+a_mx_mt^m) = \sum_{n=0}^{\infty} A_{n,m}^{a_o,\dots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} t^n,$$

where $G(z)$ is a function of $z$, $m$ is an arbitrary positive integer, $x_1,\dots,x_m$ are all complex variables and other parameters are unrestricted in general.

2.7. Differential recurrence relations. From (2.6.1), we can get the following set of 2m results :

$$(2.7.1)\quad D_{x_j}\, A_{o,m}^{a_o,\dots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = 0$$

and

$$(2.7.2)\quad a_j(n-j+1)A_{n-j+1,m}^{a_o,\dots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$= \sum_{i=1}^{m} i\; a_ix_i\; D_{x_j}\; A_{n-i+1,m}^{a_o,\dots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \quad n \geq m-1,$$

$$D_{x_j} \equiv \frac{\partial}{\partial x_j}, \quad j = 1,\dots,m.$$

2.8. Special cases. For $G(z) = z^p$, consider

(2.8.1) $$(a_o+a_1 x_1 t + \cdots + a_m x_m t^m)^p = \sum_{n=0}^{\infty} \Gamma^{a_o,\ldots,a_m}_{n,m,p} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} t^n,$$

from which we obtain

(2.8.2) $$\Gamma^{a_o,\ldots,a_m}_{n,m,p+q} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \sum_{i=0}^{n} \Gamma^{a_o,\ldots,a_m}_{n-i,m,p} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \Gamma^{a_o,\ldots,a_m}_{i,m,q} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

(2.8.3) $$\Gamma^{a_o,\ldots,a_m}_{n,m,p_1+\cdots+p_s} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \sum_{i_1+\cdots+i_s=n} \prod_{j=1}^{s} \Gamma^{a_o,\ldots,a_m}_{i_j,m,p_j} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

and

(2.8.4) $$\left(\frac{n+1}{p+1}\right) \Gamma^{a_o,\ldots,a_m}_{n+1,m,p+1} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \sum_{i=1}^{n1} i\, a_i\, x_i\, \Gamma^{a_o,\ldots,a_m}_{n-i+1,m,p} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

The following results will also follow on the lines adopted in section §2 :

$$(2.8.5)\qquad D_{x_j}\, \Gamma_{o,m,p}^{a_o,\ldots,a_m} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = 0$$

and

$$(2.8.6)\qquad a_j(n-j+1)\, \Gamma_{n-j+1,m,p}^{a_o,\ldots,a_m} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$= \sum_{i=1}^{m} i\, a_i\, x_i\, D_{x_j}\, \Gamma_{n-i+1,m,p}^{a_o,\ldots,a_m} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

$$n \geq m-1,\ j = 1,\ldots,m.$$

Again, if

$$(2.8.7)\qquad G(z) = \sum_{n=0}^{\infty} c_n\, z^n,$$

then

$$(2.8.8)\qquad A_{n,m}^{a_o,\ldots,a_m} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \sum_{i=0}^{\infty} c_i\, \Gamma_{n,m,i}^{a_o,\ldots,a_m} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}.$$

2.9. The Second Set of Polynomials. To unify the study of three general classes of polynomials considered by Gould [4], Panda [5] and Srivastava [6], Chandel [3] has introduced the polynomials defined by

$$(2.9.1)\quad (C-mxt+yt^m)^p\ G\left[\frac{r^r\ x\ t^s}{(C-mxt+yt^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} R_n^p(m,x,y,r,s,c)\ t^n,$$

where m,s are any positive integers and other parameters are unrestricted in general.

Here, in this section, we further generalize (2.9.1) and introduce polynomial systems of several variables defined by

$$(2.9.2)\quad (a_o+a_1x_1t+\ldots+a_mx_mt^m)^p\ G\left[\frac{r^r\ x_i\ t^s}{(a_o+a_1x_1t+\ldots+a_mx_mt^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} B_{n,m,p,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 & ,x_i \\ \vdots & \\ x_m & \end{bmatrix} t^n,$$

where i,r,s, $m \geq 1$ are integers $x_1,\ldots,x_m$ are complex numbers and other parameters are unrestricted in general and

$$(2.9.3)\qquad G(z) = \sum_{n=0}^{\infty} \gamma_n\ z^n \quad (\gamma_o \neq 0)$$

Therefore, we obtain

$$(2.9.4)\quad B_{n,m,p,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1, & x_i \\ \vdots & \\ x_m & \end{bmatrix}$$

$$= \sum_{j=0}^{[n/s]} \gamma_j\ r^{r_j}\ x_i^j\ \Gamma_{n-sj,m,p-rj}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

$$(2.9.5)\quad B_{n,m,p+q,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix} = \sum_{j=0}^{n} \Gamma_{n-j,m,p}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} B_{j,m,q,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix},$$

$$(2.9.6)\quad B_{n,m,p_1+\cdots+p_t,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix} = \sum_{i_1+\cdots+i_t=n} B_{i_1,m,p_1,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix} \prod_{j=2}^{t} \Gamma_{i_j,m,p}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

and

$$(2.9.7)\quad B_{n,m,p+1,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix} = \sum_{j=0}^{m} a_j x_j\, B_{n-j,m,p,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix},$$

where $x_o = 1$.

2.10. <u>Differential recurrence relations.</u> Starting with (2.9.1), we obtain the set of following 3m-1 results :

$$(2.10.1)\quad \sum_{k=0}^{m}\sum_{q=0}^{k} a_q x_q a_{k-q} x_{k-q}\,[s-(k-q)r] D_{x_j} B_{n-k,m,p,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$= a_j \sum_{\ell=0}^{m} a_\ell x_\ell\,[p-r(n-j-\ell)]\, B_{n-j-\ell,m,p,r,s}^{a_o,\ldots,a_m}\begin{bmatrix} x_1 , x_i \\ \vdots \\ x_m \end{bmatrix},$$

where $j = 1,\ldots,m$ but $j \neq i \leq m$ and $x_o = 1$,

(2.10.2) $$D_{x_j} B^{a_o,\ldots,a_m}_{o,m,p,r,s}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix} = 0, \quad j = 1,\ldots,m$$

and

(2.10.3) $$\sum_{k=0}^{m} \sum_{j=0}^{k} a_j x_j a_{k-j} x_{k-j} \left[x_i\{s-(k-j)r\}D_{x_i} + p(k-j)+k-n\right] B^{a_o,\ldots,a_m}_{n-k,m,p,r,s}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$= a_i x_i \sum_{j=0}^{m} a_j x_j \left[ps - r(n-i-j)\right] B^{a_o,\ldots,a_m}_{n-i-j,m,p,r,s}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}.$$

2.11. <u>An interesting special case.</u> Taking $\gamma_n = \frac{(-1)^n}{n!}$ in (2.9.2) we obtain

(2.11.1) $$(a_o+a_1x_1t+\ldots+a_mx_mt^m)^p \exp\left[-\frac{r^r x_i t^s}{(a_o+a_1x_1t+\ldots+a_mx_mt^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} C^{a_o,\ldots,a_m}_{n,m,p,r,s}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix} t^n,$$

from which we find that

(2.11.2) $$C^{a_o,\ldots,a_m}_{n,m,p,r,s}\begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$= \sum_{j=0}^{[n/s]} \frac{(-1)^j}{j!} r^{rj} x_i^j \, \Gamma^{a_o,\ldots,a_m}_{n-sj,m,p-rj}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

$$(2.11.3) \quad C_{n,m,p+q,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1, x_i+x_j \\ \vdots \\ x_m \end{bmatrix}$$

$$= \sum_{k=0}^{n} C_{n-k,m,p,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix} C_{k,m,q,r,s}^{q_o,\ldots,a_m} \begin{bmatrix} x_1, x_j \\ \vdots \\ x_m \end{bmatrix}$$

$$(2.11.4) \quad C_{n,m,p_1+\ldots+p_t,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1, x_1+\ldots+x_t \\ \vdots \\ x_m \end{bmatrix}$$

$$= \sum_{k_1+\ldots+k_t=n} \prod_{i=1}^{t} C_{k_i,m,p_i,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}$$

and

$$(2.11.5) \quad C_{n,m,p,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1, x_i+x_j \\ \vdots \\ x_m \end{bmatrix}$$

$$= \sum_{k=0}^{[n/s]} \frac{(-1)^k \, r^{rk} \, x_j^k}{k!} \, C_{n-ks,m,p-rk,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}.$$

Other results involving the $C_{n,m,p,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}$ defined by (2.11.1) can be derived on the lines adopted in the Sections §2.9 and §2.10.

2.12. <u>Another Special Case.</u> Taking $\gamma_n = \frac{(q)_n}{n!}$ in (2.9.2), we obtain

$$(2.12.1)\quad (a_o+\dots+a_m x_m t^m)^p \left[1 - \frac{r^r\ x_i\ t^s}{(a_o+a_1x_1t+\dots+a_mx_mt^m)^r}\right]^{-q}$$

$$= \sum_{n=0}^{\infty} E^{a_o,\dots,a_m}_{n,m,p,q,r,s} \begin{bmatrix} x_1, & x_i \\ \vdots & \\ x_m & \end{bmatrix} t^n,$$

from which we find that

$$(2.12.2)\quad E^{a_o,\dots,a_m}_{n,m,p,q,r,s} \begin{bmatrix} x_1, & x_i \\ \vdots & \\ x_m & \end{bmatrix}$$

$$= \sum_{j=0}^{[n/s]} \frac{(q)_j}{j!}\, r^{rj}\, x_i^j\, \Gamma^{a_o,\dots,a_m}_{n-sj,m,p-rj} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix},$$

$$(2.12.3)\quad E^{a_o,\dots,a_m}_{n,m,p_1+p_2,q_1+q_2,r,s} \begin{bmatrix} x_1, & x_i \\ \vdots & \\ x_m & \end{bmatrix}$$

$$= \sum_{k=0}^{n} E^{a_o,\dots,a_m}_{n-k,m,p_1,q_1,r,s} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix} E^{a_o,\dots,a_m}_{k,m,p_2,q_2,r,s} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$(2.12.4)\quad E^{a_o,\dots,a_m}_{n,m,p_1+\dots+p_t,q_1+\dots+q_t,r,s} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$= \sum_{k_1+\dots+k_t=n} \prod_{j=1}^{t} E^{a_o,\dots,a_m}_{k_j,m,p_j,q_j,r,s} \begin{bmatrix} x_1, & x_i \\ \vdots & \\ x_m & \end{bmatrix},$$

$$(2.12.5)\quad E^{a_o,\dots,a_m}_{n,m,p,q+\ell,r,s}\begin{bmatrix} x_1,x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$= \sum_{j=0}^{[n/s]} \frac{(\ell)_j}{j!}\, r^{rj}\, x^j\, E^{a_o,\dots,a_m}_{n-sj,m,p-rj,q,r,s}\begin{bmatrix} x_1,x_i \\ \vdots \\ x_m \end{bmatrix}$$

and

$$(2.12.6)\quad \sum_{j=0}^{n} \Gamma^{a_o,\dots,a_m}_{j,m,r,}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} E^{a_o,\dots,a_m}_{n-j,m,p,q+1,r,s}\begin{bmatrix} x_1,x_i \\ \vdots \\ x_m \end{bmatrix}$$

$$= r^r x^i E^{a_o,\dots,a_m}_{n-s,m,p,q+\ell,r,s}\begin{bmatrix} x_1,x_i \\ \vdots \\ x_m \end{bmatrix} + E^{a_o,\dots,a_m}_{n,m,p+r,r,s}\begin{bmatrix} x_1,x_i \\ \vdots \\ x_m \end{bmatrix}.$$

Other results involving the $E^{a_o,\dots,a_m}_{n,m,p,q,r,s}\begin{bmatrix} x_1,x_i \\ \vdots \\ x_m \end{bmatrix}$ defined by (2.12.1) can be derived on the lines adopted in the sections §2.9 and §2.10.

In the last we remark that in the above two special cases studied in Sections §2.11 and §2.12 only those results which can not be derived in general (except (2.11.2) and (2.12.2)), have been discussed. Similarly, other special cases will also follow by taking different values of $\gamma_n$ in (2.9.3).

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CHAPTER III

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## SRIVASTAVA THEOREM AND ITS BINOMIAL ANALOG

3.1. Introduction. To generalize the Carlitz theorem [1, p. 521] Srivastava [12] has recently proved the following :

Theorem. Let A(z), B(z) and $z^{-1}$ C(z) be arbitrary functions which are analytic in the neighbourhood of the origin, and assume that

$$(3.1.1) \qquad A(0) = B(0) = C'(0) = 1.$$

Define the sequence of functions $\{f_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ by means of

$$(3.1.2) \qquad A(z)\,[B(z)]^{\alpha} \exp\,(xC(z)) = \sum_{n=0}^{\infty} f_n^{(\alpha)}(x)\,\frac{z^n}{n!},$$

where $\alpha$ and x are arbitrary complex numbers independent of z.

Then, for arbitrary parameters $\lambda$ and y independent of z,

$$(3.1.3) \qquad \sum_{n=0}^{\infty} f_n^{(\alpha+\lambda n)}(x+ny)\,\frac{t^n}{n!} = \frac{A(\zeta)\,[B(\zeta)]^{\alpha} \exp\,(x\,C(\zeta))}{1-\zeta\{\lambda\,[\frac{B'(\zeta)}{B(\zeta)}] + y\,C'(\zeta)\}},$$

where

$$(3.1.4) \qquad \zeta = t\,[B(\zeta)]^{\lambda} \exp\,(y\,C(\zeta)).$$

More generally, if the function $A(z)$, $B_i(z)$ and $z^{-1}\, C_j(z)$ are analytic about the origin such that

$$(3.1.5)\qquad A(0) = B_i(0) = C'_j(0) = 1,\ i = 1,\ldots,r;\ j = 1,\ldots,s,$$

and if

$$(3.1.6)\qquad A(z) \prod_{i=1}^{r} \{[B_i(z)]^{\alpha_i}\} \exp\left(\sum_{j=1}^{s} x_j\, C_j(z)\right)$$

$$= \sum_{n=0}^{\infty} g_n^{(\alpha_1,\ldots,\alpha_r)} (x_1,\ldots,x_s) \frac{z^n}{n!},$$

then for arbitrary $\alpha$'s, $\lambda$'s , $x$'s and $y$'s independent of $z$,

$$(3.1.7)\qquad \sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n,\ldots,\alpha_r+\lambda_r n)} (x_1+ny_1,\ldots,x_s+ny_s) \frac{t^n}{n!}$$

$$= \frac{A(\zeta) \prod_{i=1}^{r} \{[B_i(\zeta)]^{\alpha_i}\} \exp\left(\sum_{j=1}^{s} x_j\, C_j(\zeta)\right)}{1 - \zeta\left\{\sum_{i=1}^{r} \lambda_i \left[\frac{B'_i(\zeta)}{B_i(\zeta)}\right] + \sum_{j=1}^{s} y_j\, C'_j(\zeta)\right\}},$$

where

$$(3.1.8)\qquad \zeta = t \prod_{i=1}^{r} \{[B_i(\zeta)]^{\lambda_i}\} \exp\left(\sum_{i=1}^{s} y_j\, C_j(\zeta)\right).$$

A number of applications of this theorem have also been given.

In this chapter, first of all we shall give some additional interesting applications of the above theorem and then we shall establish its binomial analog with interesting applications.

## 3.2 Extensions of Srivastava and Buschman results.

For $r = s = 0$, the result (14) due to Srivastava and Buschman [11] reduces to

$$\sum_{n=0}^{\infty} \binom{\alpha+2n}{n} {}_{p+q-1}F_{p+q-1} \left[ \begin{array}{l} \Delta(q,-n),\ \Delta(p-1\ ,1+\alpha+2n); \\ \Delta(p+q-1,1+\alpha+n); \end{array} \frac{q^q\,(p-1)^{p-1}}{(p+q-1)^{p+q-1}} \right] z^n$$

$$= (1-4z)^{-1/2} \left(\frac{2}{1+(1-4z)^{1/2}}\right)^{\alpha} \exp\left[ x(-z)^q \left(\frac{2}{1+(1-4z)^{1/2}}\right)^{p+2q-1}\right].$$

For $A(z) = (1-4z)^{-1/2}$, $B(z) = 2\,[1+(1-4z)^{1/2}]^{-1}$ and $C(z) = (-z)^q\, 2^{p+2q-1}\ [1+(1-4z)^{1/2}]^{-p-2q+1}$, an appeal to the above theorem gives

$$(3.2.1)\qquad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n+2n}{n}\ {}_{p+q-1}F_{p+q-1} \left[ \begin{array}{l} \Delta(q,-n),\Delta(p-1,1+\alpha+\lambda n+2n); \\ \Delta(p+q-1,1+\alpha+\lambda n+n); \end{array} \frac{q^q(p-1)^{p-1}}{(p+q-1)^{p+q-1}}\,(x+ny) \right] t^n$$

$$= \frac{(1-4\zeta)^{-1/2} \left[\frac{2}{1+(1-4\zeta)^{1/2}}\right]^{\alpha} \exp\left[ x(-\zeta)^q \left(\frac{2}{1+(1-4\zeta)^{1/2}}\right)^{p+2q-1}\right]}{1-\zeta\left[ \frac{2\lambda}{(1-4\zeta)^{1/2}\{1+(1-4\zeta)^{1/2}\}} + \bar{M}\right.}$$

$$\bar{M} = \frac{(-1)^q\, 2^{p+2q-1}\, y\, \zeta^{q-1}}{[1+(1-4\zeta)^{1/2}]^{p+2q}} \{q(1+(1-4\zeta)^{1/2}) + 2\zeta(p+2q-1)(1-4\zeta)^{-\frac{1}{2}}\}]$$

where

$$(3.2.2) \qquad \zeta = t\left[\frac{2}{1+(1-4\zeta)^{1/2}}\right]^{\lambda} \exp\left[y(-\zeta)^q\left(\frac{2}{1+(1-4\zeta)^{1/2}}\right)^{p+2q-1}\right].$$

For $r = s = 0$ we have from [11,(15)]

$$\sum_{n=0}^{\infty} z^n \binom{\alpha+2n}{n} {}_qF_q\left[\begin{matrix}\Delta(q,-n); \\ \Delta(q-p+1,1+\alpha+n),\ \Delta(p-1,-\alpha-2n);\end{matrix}\ \frac{q^q\, x}{(1-p)^{p-1}\,(q-p+1)^{1+q-p}}\right]$$

$$= (1-4z)^{-1/2}\left(\frac{2}{1+(1-4z)^{1/2}}\right)^{\alpha} \exp\left[x(-z)^q\left(\frac{2}{1+(1-4z)^{1/2}}\right)^{2q-p+1}\right],$$

$$p \leq q+1.$$

Now taking $A(z) = (1-4z)^{-1/2}$, $B(z) = 2\,[1+(1-4z)^{1/2}]^{-1}$,

$C(z) = \dfrac{(-1)^q\, z^q\, 2^{2q-p+1}}{(1+(1-4z)^{1/2})^{2q-p+1}}$ and applying the theorem we obtain

$$(3.2.3) \qquad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n+2n}{n} {}_qF_q\left[\begin{matrix}\Delta(q,-n); \\ \Delta(q-p+1,1+\alpha+\lambda n+n),\ \Delta(p-1,-\alpha-\lambda n-2n);\end{matrix}\ \frac{q^q\,(x+ny)}{(1-p)^{p-1}\,(q-p+1)^{q-p+1}}\right] t^n$$

$$= \frac{(1-4\zeta)^{-1/2}\left[\dfrac{2}{1+(1-4\zeta)^{1/2}}\right]^{\alpha} \exp\left[x(-\zeta)^q \left(\dfrac{2}{1+(1-4\zeta)^{1/2}}\right)^{2q-p+1}\right]}{1 - \zeta\left[\dfrac{2\lambda}{1-4\zeta+(1-4\zeta)^{1/2}}\right.}$$

$$+ y(-1)^q\, 2^{2q-p+1}\, \zeta^{q-1} \left\{\frac{q\{1+(1-4\zeta)^{1/2}\}+2\zeta(1-p)}{(1+(1-4\zeta)^{1/2})^{2q-p+2}(1-4\zeta)^{1/2}}\right\}\Bigg]$$

where

$$(3.2.4) \qquad \zeta = t\left[\frac{2}{1+(1-4\zeta)^{1/2}}\right]^{\lambda} \exp\left[y(-\zeta)^q \left(\frac{2}{1+(1-4\zeta)^{1/2}}\right)^{2q-p+1}\right]$$

and $p \leq q+1$.

Similarly for $r = s = 0$, the result due to Srivastava and Buschman [11,(16)] gives

$$\sum_{n=0}^{\infty} \binom{\alpha+2n}{n}\; {}_{p-1}F_{p-1}\left[\begin{matrix}\Delta(q,-n),\ \Delta(p-q-1,\alpha-n); \\ \Delta(p-1,-\alpha-2n);\end{matrix}\ \frac{(-q)^q(p-q-1)^{p-q-1}}{(p-1)^{p-1}}x\right] z^n$$

$$= (1-4z)^{-1/2}\left[\frac{2}{1+(1-4z)^{1/2}}\right]^{\alpha} \exp\left[x(-z)^q \left(\frac{2}{1+(1-4z)^{1/2}}\right)^{2q-p+1}\right],$$

$$p \geq q+1.$$

Therefore, for $A(z) = (1-4z)^{-1/2}$, $B(z) = 2\left[1+(1-4z)^{1/2}\right]^{-1}$ and $C(z) = (-1)^q\, 2^{2q-p+1}\, z^q \left[1+(1-4z)^{1/2}\right]^{-2q+p-1}$,

the main theorem gives

$$(3.2.5) \quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n+2n}{n} {}_{p-1}F_{p-1} \left[ \begin{matrix} \Delta(q,-n),\ \Delta(p-q-1,\alpha+\lambda n-n); \\ \Delta(p-1,-\alpha-\lambda n-2n); \end{matrix} \ \frac{(-q)^q (p-q-1)^{p-q-1} (x+ny)}{(p-1)^{p-1}} \right] t^n$$

$$= \frac{(1-4\zeta)^{-1/2}\, 2^{\alpha} \, [1+(1-4\zeta)^{1/2}]^{-\alpha} \exp[x(-\zeta)^q \, 2^{2q-p+1} \{1+(1-4\zeta)^{1/2}\}^{p-2q-1}]}{1-\zeta\left[\dfrac{2\lambda}{1-4\zeta+(1-4\zeta)^{1/2}} + (-1)^q \, 2^{2q-p+1} y \zeta^{q-1} \dfrac{[q\{1+(1-4\zeta)^{1/2}\}+2(2q-p+1)\zeta(1-4\zeta)^{-1/2}]}{\{1+(1-4\zeta)^{1/2}\}^{2q-p+2}}\right]}$$

where

$$(3.2.6) \quad \zeta = t\left[\frac{2}{1+(1-4\zeta)^{1/2}}\right]^{\lambda} \exp\left[y(-\zeta)^q \left(\frac{2}{1+(1-4\zeta)^{1/2}}\right)^{2q-p+1}\right],$$

and $p \geq q+1$.

For $r = s = 0$, the result (17) due to Srivastava and Buschman [11] gives

$$\sum_{n=0}^{\infty} \binom{\alpha+n}{n} {}_qF_q \left[ \begin{matrix} \Delta(q,-n); \\ \Delta(q-p,1+\alpha),\Delta(p,-\alpha-n); \end{matrix} \ \frac{q^q\, x}{(-p)^p\, (q-p)^{q-p}} \right] z^n$$

$$= (1-z)^{-\alpha-1} \exp\left[\frac{x(-z)^q}{(1-z)^{q-p}}\right],$$

where p is positive integer $\leq$ q.

Therefore for $A(z) = (1-z)^{-1}$, $B(z) = (1-z)^{-1}$ and $C(z) = \dfrac{(-z)^q}{(1-z)^{q-p}}$

an appeal to the main theorem gives

$$(3.2.7)\quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n+n}{n} t^n \; {}_qF_q\left[\begin{matrix}\Delta(q,-n);\\ \Delta(q-p,1+\lambda n+\alpha),\ \Delta(p,-\alpha-\lambda n-n);\end{matrix}\ \frac{q^q(x+ny)}{(-p)^p(q-p)^{q-p}}\right]$$

$$= \frac{(1-\zeta)^{-\alpha-1}\exp\left[\dfrac{x(-\zeta)^q}{(1-\zeta)^{q-p}}\right]}{1-\zeta\left[\dfrac{\lambda}{1-\zeta}+\dfrac{y(-1)^q\,\zeta^{q-1}(q-\zeta p)}{(1-\zeta)^{q-p+1}}\right]}$$

where

$$(3.2.8)\qquad \zeta = t\left[\frac{1}{1-\zeta}\right]^{\lambda} \exp\left[\frac{y(-\zeta)^q}{(1-\zeta)^{q-p}}\right],$$

and p is a positive integer $\leq$ q.

Again for $r = s = 0$, the result (18) due to Srivastava and Buschman [11] gives

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} \; {}_qF_q\left[\begin{matrix}\Delta(q,-n);\\ \Delta(p,-\alpha),\ \Delta(q-p,1+\alpha-n);\end{matrix}\ \frac{q^q\,x}{(-p)^p(q-p)^{q-p}}\right] z^n$$

$$= (1+z)^{\alpha}\exp\left[\frac{x(-z)^q}{(1+z)^p}\right], \quad p \geq q.$$

Therefore for $A(z) = 1$, $B(z) = (1+z)$, $C(z) = \dfrac{(-z)^q}{(1+z)^p}$,

the main theorem shows that

$$(3.2.9) \quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n}{n} {}_qF_q \left[ \begin{matrix} \Delta(q,-n); \\ \Delta(p,-\alpha-\lambda n),\ \Delta(q-p,1+\alpha+\lambda n-n); \end{matrix} \quad \frac{q^q\,(x+ny)}{(-p)^p(q-p)^{q-p}} \right] t^n$$

$$= \frac{(1+\zeta)^\alpha \exp\left[\dfrac{x(-\zeta)^q}{(1+\zeta)^p}\right]}{1-\zeta\left\{\dfrac{\lambda}{1+\zeta} + y(-1)^q\ \zeta^{q-1}\left[\dfrac{q+(q-p)\zeta}{(1+\zeta)^{p+1}}\right]\right\}},$$

where

$$(3.2.10) \quad \zeta = t(1+\zeta)^\lambda \exp\left(\frac{y(-\zeta)^q}{(1+\zeta)^p}\right) \text{ and } p \geq q.$$

Now for $r = s = 0$, [11,(19)] gives

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} {}_pF_p \left[ \begin{matrix} \Delta(p-q,-\alpha+n),\ \Delta(q,-n); \\ \Delta(p,-\alpha); \end{matrix} \quad \frac{(-q)^q\ (p-q)^{p-q}}{p^p}\ x \right] z^n$$

$$= (1+z)^\alpha \exp\left[\frac{x(-z)^q}{(1+z)^p}\right],\ p \geq q.$$

Here choosing $A(z) = 1$, $B(z) = 1+z$, $C(z) = \dfrac{(-z)^q}{(1+z)^p}$

in the main theorem, we establish

$$(3.2.11) \quad \sum_{n=0}^{\infty} t^n \binom{\alpha+\lambda n}{n} {}_pF_p \left[\begin{matrix} \Delta(p-q,-\alpha-\lambda n+n),\ \Delta(q,-n); \\ \Delta(p,-\alpha-\lambda n); \end{matrix} \ \frac{(-q)^q\ (p-q)^{p-q}\ (x+ny)}{p^p}\right]$$

$$= \frac{(1+\zeta)^{\alpha} \exp\left[\frac{x(-\zeta)^q}{(1+\zeta)^p}\right]}{1-\zeta\left[\frac{\lambda}{1+\zeta} + y(-1)^q\ \zeta^{q-1}\ \left(\frac{q+(q-p)\zeta}{(1+\zeta)^{p+1}}\right)\right]},$$

where

$$(3.2.12) \quad \zeta = t(1+\zeta)^{\lambda} \quad \exp\left[\frac{y(-\zeta)^q}{(1+\zeta)^p}\right] \text{ and } p \geq q.$$

From [11,(20)] for $r = s = 0$, we have

$$\sum_{n=0}^{\infty} \binom{\alpha-n}{n} {}_{p+q}F_{p+q} \left[\begin{matrix} \Delta(p,1+\alpha-n),\ \Delta(q,-n); \\ \Delta(p+q,1+\alpha-2n); \end{matrix} \ \frac{p^p\ q^q\ x}{(p+q)^{p+q}}\right] z^n$$

$$= (1+4z)^{-1/2}\left[\frac{2}{1+(1+4z)^{1/2}}\right]^{-\alpha-1} \exp\quad x(-z)^q\ \left(\frac{2}{1+(1+4z)^{1/2}}\right)^{q-p}].$$

Now for $A(z) = (1+4z)^{-1/2}$, $B(z) = \dfrac{1+(1+4z)^{1/2}}{2}$,

$C(z) = (-z)^q\ 2^{q-p}\ [1+(1+4z)^{1/2}\ ]^{p-q}$ an appeal to the main theorem shows that

$$(3.2.13) \quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n-1-n}{n} {}_{p+q}F_{p+q} \left[\begin{matrix} \Delta(p,\alpha+\lambda n-n),\ \Delta(q,-n); \\ \Delta(p+q,\alpha+\lambda n-2n); \end{matrix}\right.$$

$$\left.\frac{p^p\, q^q (x+ny)}{(p+q)^{p+q}}\right]\, t^n$$

$$= \frac{(1+4\zeta)^{-1/2}\left[\frac{2}{1+(1+4\zeta)^{1/2}}\right]^{-\lambda} \exp\left[x(-\zeta)^q 2^{q-p}\{1+(1+4\zeta)^{1/2}\}^{p-q}\right]}{1 - \zeta\left[\frac{2\lambda}{1+4\zeta+(1+4\zeta)^{1/2}} + \frac{(-1)^q\, 2^{q-p}\, y\, \zeta^{q-1}\left[q+q(1+4\zeta)^{1/2} + 2\zeta(p+q)\right]}{(1+4\zeta)^{1/2}\{1+(1+4\zeta)^{1/2}\}^{q-p+1}}\right]}$$

where

$$\zeta = t\left[\frac{1+(1+4\zeta)^{1/2}}{2}\right]^{\lambda} \exp\left[y(-\zeta)^q 2^{q-p}\{1+(1+4\zeta)^{1/2}\}^{p-q}\right]. \tag{3.2.14}$$

3.3 Extension of other results

Recently Chandel and Yadava [5] studied some polynomials whose one of the special case is

$$(a_o+a_1 x_1 t+\ldots+a_m x_m t^m)^p \exp\left[-\frac{r^r\, x_i\, t^s}{(a_o+a_1 x_1 t+\ldots+a_m x_m t^m)^r}\right]$$

$$= \sum_{n=0}^{\infty} C_{n,m,p,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1, x_i \\ \vdots \\ x_m \end{bmatrix} t^n,$$

where $i,s,m \geq 1$ are any integers and other parameters are unrestricted in general.

Thus for $x_i$ different from $x_1, x_2, \ldots, x_m$ and $A(z) = 1$,
$B(z) = a_o + a_1\, x_1\, z + \ldots + a_m\, x_m\, z^m$, $\alpha = p$,

$$C(z) = \frac{-r^r z^s}{(a_o+a_1 x_1 z + a_2 x_2 z^2 +\dots+ a_m x_m z^m)^r},$$

the main theorem gives

$$(3.3.1) \qquad \sum_{n=0}^{\infty} C^{a_o,\dots,a_m}_{n,m,p+\lambda n,r,s} \begin{bmatrix} x_1, x_i+ny \\ \vdots \\ x_m \end{bmatrix} t^n$$

$$= \frac{(a_o+a_1x_1\zeta+\dots+a_mx_m\zeta^m)^p \exp\left[\dfrac{-r^r x_i \zeta^s}{(a_o+a_1x_1\zeta+\dots+a_mx_m\zeta^m)^r}\right]}{1-\zeta\left[\lambda\left(\dfrac{a_1x_1+2a_2x_2\zeta+\dots+m\, a_mx_m\zeta^{m-1}}{a_o+a_1x_1\zeta+\dots+a_mx_m\zeta^m}\right) - yr^r \zeta^{s-1}\left\{\dfrac{sa_o+a_1x_1(s-r)\zeta+\dots+a_mx_m(s-mr)\zeta^m}{(a_o+a_1x_1\zeta+\dots+a_mx_m\zeta^m)^{r+1}}\right\}\right]}$$

where

$$(3.3.2) \qquad \zeta = t(a_o+a_1x_1\zeta+\dots+a_mx_m\zeta^m)^\lambda \exp\left[\frac{-r^r y \zeta^s}{(a_o+a_1x_1\zeta+\dots+a_mx_m\zeta^m)^r}\right].$$

Applying same techniques several other applications can also be shown.

## 3.4 A binomial analog of Srivastava theorem.

In this section, we shall give a following binomial analog of Srivastava theorem [12] :

**Theorem.** Let $A(z)$, $B(z)$ and $z^{-1}$ $C(z)$ be arbitrary functions

which are analytic in the neighbourhood of the origin, and assume that

(3.4.1) $\quad A(0) = B(0) = C'(0) = 1.$

Define the sequence of functions $\{g_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$ by means of

(3.4.2) $$A(z)\ B(z)^{\alpha}[1-xC(z)]^{-\beta} = \sum_{n=0}^{\infty} g_n^{(\alpha,\beta)}(x)\frac{z^n}{n!},$$

where $\alpha,\beta$ and $x$ are arbitrary complex numbers independent of $z$. Then, for arbitrary parameters $\lambda$ and $\mu$ independent of $z$,

(3.4.3) $$\sum_{n=0}^{\infty} g_n^{(\alpha+\lambda n,\beta+\mu n)}(x)\frac{t^n}{n!}$$

$$= \frac{A(\zeta)\ [B(\zeta)]^{\alpha}\ [1-xC(\zeta)]^{-\beta}}{1-\zeta\{\lambda[B'(\zeta)/B(\zeta)] + \frac{x\mu\ C'(\zeta)}{1-x\ C(\zeta)}\}}$$

where

(3.4.4) $$\zeta = t\,[B(\zeta)]^{\lambda}[1 - x\,C(\zeta)]^{-\mu}.$$

<u>Generalization</u>. If the functions $A(z)$, $B_i(z)$ and $z^{-1}\ C_j(z)$ are analytic about the origin such that

(3.4.5) $\quad A(0) = B_i(0) = C_j'(0),\ i = 1,\ldots,r;\ j = 1,\ldots,s,$

and if

(3.4.6) $$A(z)\prod_{i=1}^{r}[B_i(z)]^{\alpha_i}\prod_{j=1}^{s}[1-x_j\,C_j(z)]^{-\beta_j}$$

$$= \sum_{n=0}^{\infty} g_n^{(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_s)}(x_1,\ldots,x_s)\frac{z^n}{n!}$$

then for arbitrary $\alpha$'s, $\beta$'s, $\lambda$'s and $\mu$'s independent of z,

(3.4.7) $$\sum_{n=0}^{\infty} g_n^{(\alpha_1+\lambda_1 n,\ldots,\alpha_r+\lambda_r n;\beta_1+\mu_1 n,\ldots,\beta_s+\mu_s n)}(x_1,\ldots,x_s)\frac{t^n}{n!}$$

$$= \frac{A(\zeta)\prod_{i=1}^{r}[B_i(\zeta)]^{\alpha_i}\prod_{j=1}^{s}[1-x_j\, C_j(\zeta)]^{-\beta_j}}{1-\zeta\left[\sum_{i=1}^{r}\lambda_i\left(\frac{B_i'(\zeta)}{B_i(\zeta)}\right)+\sum_{j=1}^{s}\frac{x_j\,\mu_j\,C_j'(\zeta)}{1-x_j\,C_j(\zeta)}\right]}$$

where

(3.4.8) $$\zeta = t\prod_{i=1}^{r}(B_i(\zeta))^{\lambda_i}\prod_{j=1}^{s}[1-x_j\, C_j(\zeta)]^{-\mu_j}.$$

3.5. Proof of the theorem. By Taylor's theorem, (3.4.2) gives

(3.5.1) $$g_n^{(\alpha,\beta)}(x) = D_z^n\{A(z)[B(z)]^{\alpha}[1-x\,C(z)]^{-\beta}\}\Big|_{z=0}$$

Therefore

(3.5.2) $$g_n^{(\alpha+\lambda n,\beta+\mu n)}(x) = D_z^n\{f(z)\,[\Phi(z)]^n\}\Big|_{z=0}$$

where

(3.5.3) $$f(z) = A(z)\,[B(z)]^{\alpha}\,[1-x\,C(z)]^{-\beta},$$

$$\Phi(z) = [B(z)]^{\lambda}\,[1-x\,C(z)]^{-\mu}.$$

Whence from (3.5.2), we have

$$(3.5.4) \qquad \sum_{n=0}^{\infty} g_n^{(\alpha+\lambda n,\beta+\mu n)}(x) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} D_z^n \{f(z)[\phi(z)]^n\}\Big|_{z=0}$$

Now we apply Lagrange's expansion in the form [9,p.146]

$$(3.5.5) \qquad \sum_{n=0}^{\infty} \frac{t^n}{n!} D_z^n \{f(z)[\Phi(z)]^n\}\Big|_{z=0} = \frac{f(\zeta)}{1-t\,\Phi'(\zeta)},$$

where the functions $f(z)$ and $\Phi(z)$ are analytic about the origin and $\zeta$ is given by

$$(3.5.6) \qquad \zeta = t\,\Phi(\zeta), \quad \Phi(0) \neq 0.$$

Thus the generating function (3.4.3) follows radily from (3.5.4) under the constraints (3.4.1) and (3.4.2).

The derivation of the multivariable (and multiparameters) generating function (3.4.7) runs parallel to that of (3.4.3) therefore here we skip the details involved.

3.6 Applications. Chandel and Bhargava [3,4] have studied the polynomials defined by

$$(1-z)^{-c}\left[1 - \frac{xz^s}{(1-z)^r}\right]^{-b} = \sum_{n=0}^{\infty} \Gamma_n^{(b,c)}(x,r,s)\, z^n.$$

Therefore, taking $A(z) = 1$, $B(z) = (1-z)^{-1}$, $C(z) = \frac{z^s}{(1-z)^r}$, $\alpha = c$ and $\beta = b$, we get from the above theorem

$$(3.6.1) \qquad \sum_{n=0}^{\infty} \Gamma_n^{(b+\mu n,c+\lambda n)}(x,r,s)\, t^n$$

$$= \frac{(1-\zeta)^{-c+1} \left[1 - \frac{x\zeta^s}{(1-\zeta)^r}\right]^{-b}}{1-\zeta-\zeta\lambda - x\mu \left[\frac{s\zeta^r + (r-s)\ \zeta^{s+1}}{(1-\zeta)^r - x\zeta^s}\right]}$$

where

$$(3.6.2) \qquad \zeta = t(1-\zeta)^{-\lambda} \left[1 - \frac{x\zeta^s}{(1-\zeta)^r}\right]^{-\mu}.$$

Chandel and Bhargava [3,(2.4)] gave the relation

$$\sum_{n=0}^{\infty} A_n^{(b,c)}(x,r,s)\ z^n = (1+z)^{-c} \left[1-xz^s(1+z)^{r-s}\right]^b .$$

Thus choosing $A(z) = 1$, $B(z) = (1+z)^{-1}$, $C(z) = z^s(1+z)^{r-s}$, $\alpha = c$ and $\beta = -b$ in the main theorem, we obtain

$$(3.6.3) \qquad \sum_{n=0}^{\infty} A_n^{(b+\mu n, c+\lambda n)}(x,r,s)\ t^n$$

$$= \frac{(1+\zeta)^{-c} \left[1-x\ \zeta^s(1+\zeta)^{r-s}\right]^b}{1+\zeta \left[\frac{\lambda}{1+\zeta} + \frac{\mu x\ \zeta^{s-1}\ (1+\zeta)^{r-s-1}(s+r\zeta)}{1-x\ \zeta^s(1+\zeta)^{r-s}}\right]}$$

where

$$(3.6.4) \qquad \zeta = t(1+\zeta)^{-\lambda} \left[1-x\ \zeta^s(1+\zeta)^{r-s}\right]^{\mu} .$$

For Gegenbauer polynomials, we have

$$\sum_{n=0}^{\infty} C_n^{\nu}(x)\ z^n = (1-2xz+z^2)^{-\nu} = (1+z^2)^{-\nu} \left[1 - \frac{2xz}{1+z^2}\right]^{-\nu} .$$

Hence taking $\beta = \alpha = \nu$, $\mu = \lambda$, $A(z) = 1$, $B(z) = (1+z^2)^{-1}$

$C(z) = \dfrac{2z}{1+z^2}$, and applying the theorem, we establish

$$(3.6.5) \qquad \sum_{n=0}^{\infty} C_n^{\nu+\lambda n}(x)t^n = \frac{(1+\zeta^2-2x\zeta)^{-\nu}}{1+\frac{2\zeta\lambda}{1+\zeta^2}\left(\zeta - \frac{(1-\zeta^2)x}{1+\zeta^2-2x\zeta}\right)},$$

where

$$(3.6.6) \qquad \zeta = t(1+\zeta^2-2x\zeta)^{-\lambda}.$$

Gould [7] considered the generalized Humbert polynomials defined by

$$\sum_{n=0}^{\infty} P_n(m,x,y,p,C)\ t^n = (C-mxt + yt^m)^p$$

$$= [C+yt^m]^p \left[1 - \frac{mxt}{C+yt^m}\right]^p .$$

Thus taking $\beta = \alpha = -p$, $\mu = \lambda$, $A(z) = 1$, $B(z) = (C+yz^m)^{-1}$, $C(z) = \frac{mz}{C+yz^m}$ in the main theorem, we derive

$$(3.6.7) \qquad \sum_{n=0}^{\infty} P_n(m,x,y,p+\lambda n,C)\ t^n$$

$$= \frac{(C-mx\zeta+y\zeta^m)^p}{1 + \frac{m\zeta\lambda}{C+y\zeta^m}\left[-y\ \zeta^{m-1} + \frac{x(C-my\zeta^m + y\zeta^m)}{C-mx\zeta+y\zeta^m}\right]}$$

where

$$(3.6.8) \qquad \zeta = t\ [C + y\ \zeta^m - xm\ \zeta\ ]^\lambda .$$

Chandel and Yadava [5,(7.1)] studied the polynomials defined by

$$(a_o+a_1x_1t+\ldots+a_mx_mt^m)^p \left[1-\frac{r^r x_i t^s}{(a_o+a_1x_1t+\ldots+a_mx_mt^m)^r}\right]^{-q}$$

$$= \sum_{n=0}^{\infty} E_{n,m,p,q,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1 & ,x_i \\ \vdots & \\ x_m & \end{bmatrix} t^n ,$$

where m, s are arbitrary positive integers, $x_1,\ldots,x_m$, $x_i$ are all complex numbers and other parameters are unrestricted in general.

Taking $A(z) = 1$, $B(z) = a_o+a_1x_1z+\ldots+a_mx_mz^m$,

$$\alpha = p,\ \beta = q,\ C(z) = \frac{r^r z^s}{(a_o+a_1x_1z+\ldots+a_mx_mz^m)^r}$$

and $x_i$ different from $x_1,\ldots,x_m$, and making an appeal to the above theorem, we get

$$(3.6.9)\quad \sum_{n=0}^{\infty} E_{n,m,p+\lambda n,q+\mu n,r,s}^{a_o,\ldots,a_m} \begin{bmatrix} x_1,x_i \\ \vdots \\ x_m \end{bmatrix} t^n$$

$$= \frac{(a_o+a_1x_1\zeta+\ldots+a_mx_m\zeta^m)^{p+1}\left[1-\frac{r^r x_i \zeta^s}{(a_o+a_1x_1\zeta+\ldots+a_mx_m\zeta^m)^r}\right]^{-q}}{\{a_o+a_1x_1\zeta(1-\lambda)+\ldots+a_mx_m(1-m\lambda)\zeta^m\} - \frac{\mu r^r x_i \zeta^s [a_os+(s-r)a_1x_1\zeta+\ldots+(s-mr)a_mx_m\zeta^m]}{(a_o+a_1x_1\zeta+\ldots+a_mx_m\zeta^m)^r - r^r x_i \zeta^s}}$$

where

(3.6.10) $\quad \zeta = t\,[a_o+a_1x_1\zeta+\ldots+a_mx_m\zeta^m]^\lambda [1-\dfrac{r^r\, x_i\, \zeta^s}{(a_o+a_1x_1\zeta+\ldots+a_mx_m\zeta^m)^r}]^{-\mu}$

Recently Chandel [2,(2.5)] has obtained the following relation for generalized Stirling numbers

$$\sum_{n=0}^{\infty} s^{(\alpha,k)}(n,m,r)\,\frac{z^n}{n!} = \frac{(-1)^m}{m!}\,[1-(k-1)z]^{-\frac{\alpha}{(k-1)}}\,[1-\{1-(k-1)z\}^{-\frac{r}{k-1}}]^m$$

$k \neq 1.$

Taking $A(z) = \dfrac{(-1)^m}{m!}$, $B(z) = [1-(k-1)z]^{-1/(k-1)}$,

$\beta = -m$, $x = 1$, $C(z) = \{1-(k-1)z\}^{-\frac{r}{(k-1)}}$ in the main theorem, we get

(3.6.11) $\quad \displaystyle\sum_{n=0}^{\infty} s^{(\alpha+n\lambda,k)}(n,m+\mu n,r)\,\frac{t^n}{n!}$

$$= \frac{(-1)^m}{m!}\cdot\frac{[1-(k-1)\zeta]^{-\alpha/(k-1)}\,[1-\{1-(k-1)\zeta\}^{-\frac{r}{(k-1)}}]^m}{1-\zeta\,[\lambda\{1-(k-1)\zeta\}^{-1} - \dfrac{\mu r\,[1-(k-1)\zeta]^{-(r+k-1)/(k-1)}}{1-[1-(k-1)\zeta]^{-r/(k-1)}}]},$$

where

(3.6.12) $\quad \zeta = t\;[1-(k-1)\zeta]^{-\lambda/(k-1)}\,[1-\{1-(k-1)\zeta\}^{-r/(k-1)}]^\mu .$

Replacing $\delta$ by $(-\alpha)$ and $\lambda$ by $\beta$, the result due to Shrivastava and Singh [10, (3.4)] reduces to

$$[1 + \frac{(x-1)}{2} z]^{\alpha} \ [1 - \frac{y(x-1)z}{2-(1-x)z}]^{-\beta}$$

$$= \sum_{n=0}^{\infty} \frac{n!}{(-\alpha)_n} [(1-y)z]^n P_n^{(-n,\alpha-n)}(x) \ P_n^{(-\alpha-1,-\beta-n)} (\frac{1+y}{1-y}).$$

Now taking $A(z) = 1$, $B(z) = [1+ \frac{(x-1)}{2} z]$, $C(z) = \frac{(x-1)z}{2-(1-x)z}$

and making an appeal to the theorem, we derive

$$(3.6.13) \quad \sum_{n=0}^{\infty} \frac{n!}{(-\alpha-\lambda n)_n} [(1-y)z]^n P_n^{(-n,\alpha+\lambda n-n)}(x)$$

$$P_n^{(-\alpha-\lambda n-1,-\beta-\mu n-n)} (\frac{1+y}{1-y})$$

$$= \frac{[1 + \frac{(x-1)}{2} \zeta]^{\alpha} \ [1 - \frac{y(x-1) \ \zeta}{2-(1-x)\zeta}]^{-\beta}}{1 - \zeta \frac{(x-1)}{2+(x-1)\zeta} \{\lambda + \frac{2\mu y}{2+(x-1) \ \zeta \ (1-y)}\}}$$

where

$$(3.6.14) \qquad \zeta = t \ [1 + (\frac{x-1}{2}) \ \zeta]^{\lambda} \ [1 - y \frac{(x-1)\zeta}{2-(1-x)\zeta}]^{-\mu}.$$

3.7 Application of multiparameters (and multivariables) generating function.

For Lauricella polynomials [8,p.113] it is readily observed that

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} F_D^{(s)} [-n,\beta_1,\ldots,\beta_s; \alpha; \gamma_1,\ldots,\gamma_s] z^n$$

$$= (1-z)^{-\alpha} \prod_{j=1}^{s} (1 + \frac{\gamma_j z}{1-z})^{-\beta_j} , \quad |z| < 1.$$

Now taking $A(z) = 1$, $B_1(z) = (1-z)^{-1}$, $\alpha_1 = \alpha$, $r = 1$ $x_j = \gamma_j$, $C_j(z) = \frac{-z}{1-z}$, in the multivariable (and multiparameters) theorem (3.4.6) to (3.4.7), we obtain the result due to Srivastava [12,(3.24)] who has obtained it by quite different substitutions in his theorem.

We have the following result due to Exton [6,(3.2)]

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} F_D^{(s)} (-n, \beta_1, \ldots, \beta_s; -1+\alpha-n; x_1, \ldots, x_s) z^n$$

$$= (1+z)^{\alpha} (1+x_1 z)^{-\beta_1} \ldots (1+x_s z)^{-\beta_s} .$$

Choosing $A(z) = 1$, $B_1(z) = 1+z$, $\alpha_1 = \alpha$, $r = 1$ and $C_j(z) = -z$, $j = 1, \ldots, s$, and making an appeal to the multivariable theorem, we derive

$$(3.7.1) \quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda_1 n}{n} F_D^{(s)}(-n, \beta_1+\mu_1 n, \ldots, \beta_s+\mu_s n; -1+\alpha+(\lambda_1-1)n; x_1, \ldots, x_s) z^n$$

$$= \frac{(1+\zeta)^{\alpha} \prod_{j=1}^{s} [1+x_j \zeta]^{-\beta_j}}{1-\zeta \left[ \frac{\lambda_1}{(1+\zeta)} - \sum_{j=1}^{s} \frac{x_j \mu_j}{1+x_j \zeta} \right]},$$

where

$$(3.7.2) \quad \zeta = t(1+\zeta)^{\lambda_1} \prod_{j=1}^{s} (1+x_j \zeta)^{-\mu_j} .$$

In the last we remark that a number of additional applications of our theorem can also be given.

## 3.8 Some additional applications

In this section, we further give some additional applications of this theorem to extend some of the results due to Srivastava and Buschman [11].

For $r = 1$, $s = 0$ the result due to Srivastava and Buschman [11,(14)] reduces to

$$\sum_{n=0}^{\infty} \binom{\alpha+2n}{n} {}_{p+q}F_{p+q-1}\left[\begin{matrix}\beta,\ \Delta(q-n),\ \Delta(p-1,1+\alpha+2n); \\ \Delta(p+q-1,1+\alpha-n);\end{matrix}\ \frac{q^q(p-1)^{p-1}\,x}{(p+q-1)^{p+q-1}}\right] z^n$$

$$= (1-4z)^{-1/2}\left[\frac{2}{(1+(1-4z)^{1/2}}\right]^{\alpha}\left[1-x(-z)^q\left(\frac{2}{1+(1-4z)^{1/2}}\right)^{p+2z-1}\right]^{-\beta}.$$

Choosing $A(z) = (1-4z)^{-1/2}$, $B(z) = \dfrac{2}{1+(1-4z)^{1/2}}$,

$$C(z) = \frac{(-z)^q\ 2^{p+2q-1}}{\{1+(1-4z)^{1/2}\}^{p+2q-1}}$$

and making an appeal to the above theorem, we obtain

$$(3.8.1)\quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n+2n}{n} {}_{p+q}F_{p+q-1}\left[\begin{matrix}\beta+\mu n,\ \Delta(q,-n),\Delta(p-1,1+\alpha+2n+\lambda n); \\ \Delta(p+q-1,1+\alpha+n+\lambda n);\end{matrix}\ \frac{q^q(p-1)^{p-1}\,x}{(p+q-1)^{p+q-1}}\right] t^n$$

$$= \frac{(1-4\zeta)^{-1/2}\left[\dfrac{2}{1+(1-4\zeta)^{1/2}}\right]^{\alpha}\left[1-x(-\zeta)^q\left(\dfrac{2}{1+(1-4\zeta)^{1/2}}\right)^{p+2q-1}\right]^{-\beta}}{1-\dfrac{\zeta}{[1+(1-4\zeta)^{1/2}]}\left[\dfrac{2\lambda}{(1-4\zeta)^{1/2}}+\bar{N}\right.}$$

*where*

$$\ddot{N} = \frac{\mu x(-1)^q\, 2^{p+2q-1}\, \zeta^{q-1}\, [q\{1+(1-4\zeta)^{1/2}\}+2\zeta(p+2q-1)\, (1-4\zeta)^{-\frac{1}{2}}]}{[\{1+(1-4\zeta)^{1/2}\}^{p+2q-1} - x(-\zeta)^q\, 2^{p+2q-1}\}]}]$$

*and*

$$(3.8.2) \quad \zeta = t\,[\frac{2}{1+(1-4\zeta)^{1/2}}]^{\lambda}\,[1-x(-\zeta)^q(\frac{2}{1+(1-4\zeta)^{1/2}})^{p+2q-1}]^{-\mu} .$$

For $r = 1$, $s = 0$ the result due to Srivastava and Buschman [11,(15)] reduces to

$$\sum_{n=0}^{\infty} z^n \binom{x+2n}{n}\ {}_{q+1}F_q \left[ \begin{matrix} \beta,\ \Delta(q,-n); \\ \Delta(q-p+1,1+\alpha+n),\ \Delta(p-1,-\alpha-2n); \end{matrix} \ \frac{q^q\, x}{(1-p)^{p-1}\, (q-p+1)^{q-p+1}} \right]$$

$$= (1-4z)^{-1/2}\ [\frac{2}{1+(1-4z)^{1/2}}]^{\alpha}\ [1-x(-z)^q(\frac{2}{1+(1-4z)^{1/2}})^{2q-p+1}]^{-\beta} .$$

Thus in the main theorem, taking

$$A(z) = (1-4z)^{-1/2} ,\quad B(z) = \frac{2}{1+(1-4z)^{1/2}} \quad \text{and}$$

$C(z) = 2^{2q-p+1}(-z)^q\ [1+(1-4z)^{1/2}]^{p-2q-1}$ , we establish

$$(3.8.3) \quad \sum_{n=0}^{\infty} t^n \binom{\alpha+\lambda n+2n}{n}\ {}_{q+1}F_q \left[ \begin{matrix} \beta+\mu n,\ \Delta(q,-n); \\ \Delta(q-p+1,1+\alpha+n+\lambda n),\Delta(p-1,-\alpha-2n-\lambda n)\cdot \end{matrix} \ \frac{q^q\, x}{(1-p)^{p-1}\, (q-p+1)^{q-p+1}} \right]$$

$$= \frac{(1-4\zeta)^{-1/2}\ \left[\frac{2}{1+(1-4\zeta)^{1/2}}\right]^{\alpha}\left[1-x(-\zeta)^{q}\left(\frac{2}{1+(1-4\zeta)^{1/2}}\right)^{2q-p+1}\right]^{-\beta}}{1-\zeta\left[\frac{2\lambda}{(1-4\zeta)^{1/2}\left[1+(1-4\zeta)^{1/2}\right]}\right. + \frac{x\mu\ (-1)^{q}\ 2^{2q-p+1}\left[1+(1-4\zeta)^{1/2}\right]^{p-2q-2}\left[q\zeta^{q-1}\{1+(1-4\zeta)^{\frac{1}{2}}\}-\frac{2\zeta^{q}(p-2q-1)}{(1-4\zeta)^{1/2}}\right]}{1-x\ 2^{2q-p+1}(-\zeta)^{q}\left[1+(1-4\zeta)^{1/2}\right]^{p-2q-1}}\left.\right]}$$

where

$$(3.8.4)\qquad \zeta = t\left[\frac{2}{1+(1-4\zeta)^{1/2}}\right]^{\lambda}\left[1-x\ 2^{2q-p+1}(-\zeta)^{q}\{1+(1-4\zeta)^{\frac{1}{2}}\}^{p-2q-1}\right]^{-\mu}$$

and $p \leq q+1$.

For $r = 1$, $s = 0$, the result due to Srivastava and Buschman [11,(16)] gives

$$\sum_{n=0}^{\infty}\binom{\alpha+2n}{n}\ {}_pF_{p-1}\left[\begin{matrix}\beta,\ \Delta(q,-n),\ \Delta(p-q-1,\alpha-n);\\ \Delta(p-1,-\alpha-2n);\end{matrix}\ \frac{(-q)^{q}(p-q-1)^{p-q-1}}{(p-1)^{p-1}}x\right]z^{n}$$

$$= (1-4z)^{-1/2}\left(\frac{2}{1+(1-4z)^{1/2}}\right)^{\alpha}\left[1-x(-z)^{q}\left(\frac{2}{1+(1-4z)^{1/2}}\right)^{2q-p+1}\right]^{-\beta},$$

$p \geq q+1$.

Now choosing $A(z) = (1-4z)^{-1/2}$, $B(z) = \frac{2}{1+(1-4z)^{1/2}}$,

$C(z) = \frac{(-1)^{q}\ z^{q}\ 2^{2q-p+1}}{\{1+(1-4z)^{1/2}\}^{2q-p+1}}$ and applying the theorem, we get

$$(3.8.5)\qquad \sum_{n=0}^{\infty}\binom{\alpha+\lambda n+2n}{n}\ {}_pF_{p-1}\left[\begin{matrix}\beta+\mu n, \Delta(q,-n)\ \Delta(p-q-1,\alpha-n+\lambda n)\\ \Delta(p-1,-\alpha-2n-\lambda n);\end{matrix}\ \frac{(-q)^{q}(p-q-1)^{p-q-1}}{(p-1)^{p-1}}x\right]t^{n}$$

$$= \frac{(1-4\zeta)^{-1/2}\left[\frac{2}{1+(1-4\zeta)^{1/2}}\right]^{\alpha}\left[1-x(-\zeta)^{q}\left(\frac{2}{1+(1-4\zeta)^{1/2}}\right)^{2q-p+1}\right]^{-\beta}}{1-\zeta\left[\lambda\left\{\frac{2}{(1-4\zeta)^{1/2}\,[1+(1-4\zeta)^{1/2}]}\right\}+\right.}$$

$$\left.+\ \mu x\left\{\frac{(-1)^{q}\,2^{2q-p+1}\,\zeta^{q-1}[1+(1-4\zeta)^{1/2}]^{p-2q-2}\left[q\{1+(1-4\zeta)^{\frac{1}{2}}\}-\frac{2\zeta(p-2q-1)}{(1-4\zeta)^{1/2}}\right]}{1-(-1)^{q}\,x\,2^{2q-p+1}\,\zeta^{q}\{1+(1-4\zeta)^{1/2}\}^{p-2q-1}}\right\}\right]$$

where

$$\zeta = t\left[\frac{2}{1+(1-4\zeta)^{1/2}}\right]^{\lambda}\left[1-\frac{x(-1)^{q}\,\zeta^{q}\,2^{2q-p+1}}{\{1+(1-4\zeta)^{1/2}\}^{2q-p+1}}\right]^{-\mu} \tag{3.8.6}$$

and $p \geq q+1$.

For $r = 1$, $s = 0$, the result [11,(17)] gives

$$\sum_{n=0}^{\infty}\binom{\alpha+n}{n}\ {}_{q+1}F_{q}\left[\begin{matrix}\beta,\ \Delta(q,-n); \\ \\ \Delta(q-p,\alpha+1),\Delta(p,-\alpha-n);\end{matrix}\ \frac{q^{q}\,x}{(-p)^{p}(q-p)^{q-p}}\right] z^{n}$$

$$= (1-z)^{-\alpha-1}\left[1-\frac{x(-z)^{q}}{(1-z)^{q-p}}\right]^{-\beta},\quad p \leq q.$$

Therefore selecting $A(z) = (1-z)^{-1}$, $B(z) = (1-z)^{-1}$,

$C(z) = \dfrac{(-z)^{q}}{(1-z)^{q-p}}$ in the main theorem, we derive

$$\sum_{n=0}^{\infty}\binom{\alpha+\lambda n+n}{n}\ {}_{q+1}F_{q}\left[\begin{matrix}\beta+\mu n,\Delta(q-n); \\ \\ \Delta(q-p,1+\alpha+\lambda n),\Delta(p,-\alpha-\lambda n-n);\end{matrix}\ \frac{q^{q}\,x}{(-p)^{p}(q-p)^{q-p}}\right] t^{n} \tag{3.8.7}$$

$$= \frac{(1-\zeta)^{-\alpha}\ [1 - \frac{x(-\zeta)^q}{(1-\zeta)^{q-p}}]^{-\beta}}{1-\zeta\,[\lambda+1+\mu x\,\{\frac{(-1)^q\ \zeta^{q-1}(q-p\zeta)}{(1-\zeta)^{q-p}-x(-\zeta)^q}\}]}$$

where

$$(3.8.8) \qquad \zeta = t(1-\zeta)^{-\lambda}\,[1 - \frac{x(-\zeta)^q}{(1-\zeta)^{q-p}}]^{-\mu}$$

and $p \leq q$.

For $r = 1$, $s = 0$, [11,(18)] reduces to

$$\sum_{n=0}^{\infty} \binom{\alpha}{n}\ {}_{q+1}F_q \left[ \begin{matrix} \beta,\ \Delta(q,-n); \\ \Delta(p,-\alpha),\Delta(q-p,1+\alpha-n); \end{matrix} \ \frac{q^q\ x}{(-p)^p(q-p)^{q-p}} \right] z^n$$

$$= (1+z)^{\alpha}\ [1 - \frac{x(-z)^q}{(1+z)^p}]^{-\beta}, \quad p \leq q.$$

Therefore for $A(z) = 1$, $B(z) = (1+z)$, $C(z) = \frac{(-z)^q}{(1+z)^p}$, the theorem gives

$$(3.8.9) \quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n}{n}\ {}_{q+1}F_q \left[ \begin{matrix} \beta+\mu n,\Delta(q,-n); \\ \Delta(p,-\alpha-\lambda n),\Delta(q-p,1+\alpha+\lambda n-n); \end{matrix} \ \frac{q^q\ x}{(-p)^p(q-p)^{q-p}} \right] t^n$$

$$= \frac{(1+\zeta)^{\alpha+1}\ [1 - \frac{x(-\zeta)^q}{(1+\zeta)^p}]^{-\beta}}{1-\zeta\,[\lambda-1 + \frac{x\mu\ (-1)^q\ \zeta^{q-1}\{q+(q-p)\zeta\}}{(1+\zeta)^p-x(-1)^q\ \zeta^q}]}$$

where

$$(3.8.10) \qquad \zeta = t\ [1+\zeta]^{\lambda}\,[1-x\,\frac{(-\zeta)^q}{(1+\zeta)^p}]^{-\mu}$$

and $p \leq q$.

For $r = 1$, $s = 0$, [11,(19)] reduces to

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} {}_{p+1}F_p \left[ \begin{matrix} \beta, \Delta(p-q,-\alpha+n), \Delta(q,-n); \\ \Delta(p,-\alpha); \end{matrix} \quad \frac{(-q)^q (p-q)^{p-q}}{p^p} x \right] z^n$$

$$= (1+z)^{\alpha} \left[1 - \frac{x(-z)^q}{(1+z)^p}\right]^{-\beta}.$$

Whence choosing $A(z) = 1$, $B(z) = (1+z)$, $C(z) = \dfrac{(-1)^q z^q}{(1+z)^p}$ in the theorem, we establish

$$(3.8.11) \quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n}{n} {}_{p+1}F_p \left[ \begin{matrix} \beta+\mu n, \Delta(p-q,-\alpha-\lambda n+n), \Delta(q,-n); \\ \Delta(p,-\alpha-\lambda n); \end{matrix} \quad \frac{(-q)^q (p-q)^{p-q}}{p^p} \right] t^n$$

$$= \frac{(1+\zeta)^{\alpha+1} \left[1 - \dfrac{x(-\zeta)^q}{(1+\zeta)^p}\right]^{-\beta}}{1-\zeta \left[\lambda-1+ \dfrac{\mu x(-1)^q \zeta^{q-1} \{q+(q-p)\zeta\}}{(1+\zeta)^p - x(-\zeta)^q}\right]},$$

where

$$(3.8.12) \qquad \zeta = t\,[1+\zeta]^{\lambda} \left[1-x \frac{(-1)^q \zeta^q}{(1+\zeta)^p}\right]^{-\mu} \text{ and } p \geq q.$$

For $r = 1$, $s = 0$ the result [11,(20)] gives

$$\sum_{n=0}^{\infty} \binom{\alpha-1-n}{n} {}_{p+q+1}F_{p+q} \left[ \begin{matrix} \beta, \Delta(p,\alpha-n), \Delta(q,-n); \\ \Delta(p+q,\alpha-2n); \end{matrix} \quad \frac{p^p q^q x}{(p+q)^{p+q}} \right] z^n$$

$$= (1+4z)^{-1/2} \left[\frac{2}{1+(1+4z)^{1/2}}\right]^{-\alpha} \left[1-x(-z)^q \left(\frac{2}{1+(1+4z)^{1/2}}\right)^{q-p}\right]^{-\beta}.$$

Therefore, choosing $A(z) = (1+4z)^{-1/2}$, $B(z) = \frac{1+(1+4z)^{1/2}}{2}$,

$C(z) = (-z)^q \left[\frac{1+(1+4z)^{1/2}}{2}\right]^{p-q}$ and making an appeal to the main theorem, we obtain

$$(3.8.13)\quad \sum_{n=0}^{\infty} \binom{\alpha+\lambda n-1-n}{n} {}_{p+q+1}F_{p+q}\left[\begin{matrix}\beta+\mu n, \Delta(p,\alpha+\lambda n-n), \Delta(q,-n);\\ \Delta(p+q,\alpha+\lambda n-2n);\end{matrix}\ \frac{p^p\, q^q\, x}{(p+q)^{p+q}}\right] t^n$$

$$= \frac{(1+4\zeta)^{-1/2}\left[\frac{2}{1+(1+4\zeta)^{1/2}}\right]^{-\alpha}\left[1-x(-\zeta)^q\left(\frac{2}{1+(1+4\zeta)^{1/2}}\right)^{q-p}\right]^{-\beta}}{1-\zeta\left[\frac{2\lambda}{(1+4\zeta)^{1/2}\,[1+(1+4\zeta)^{1/2}]} + \frac{(-1)^q\mu x\,[1+(1+4\zeta)^{1/2}]^{p-q-1}\left[\frac{2\zeta^q}{(1+4\zeta)^{1/2}}+q\zeta^{q-1}\{1+(1+4\zeta)^{1/2}\}\right]}{2^{p-q}\left[1-x(-\zeta)^q\left\{\frac{1+(1+4\zeta)^{1/2}}{2}\right\}^{p-q}\right]}\right]}$$

where

$$(3.8.14)\quad \zeta = t\left[\frac{1+(1+4\zeta)^{\frac{1}{2}}}{2}\right]^{\lambda}\left[1-x(-\zeta)^q\left\{\frac{1+(1+4\zeta)^{1/2}}{2}\right\}^{p-q}\right]^{-\mu}$$

and p,q are positive integers.

Similarly applying the same techniques several other applications of the above theorem can be established.

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By induction, we can easily prove

$$(4.2.1)\quad \Omega_x^n(x^{\alpha k-\lambda}) = k^n(\alpha)_n\; x^{\alpha k-\lambda+nk}$$

and

$$(4.2.2)\quad \frac{1}{\Omega_x^n}\left(\frac{1}{x^{(\alpha-1)k+\lambda}}\right) = \frac{(-1)^n}{(\alpha)_n\,(k)^n\,x^{\alpha k-k+\lambda+nk}}\;.$$

Now making an appeal to the above properties of the operator, we establish the following result :

$$(4.2.3)\quad \left(1+\frac{\Omega_x}{w_{1,y}}\right)^{-\beta_1}\cdots\left(1+\frac{\Omega_x}{w_{\nu,y}}\right)^{-\beta_\nu}\exp\left(-\frac{\Omega_x\,w_{\nu+1,x}}{w_{\nu+1,y}}\right)\cdots$$

$$\exp\left(\frac{-\Omega_x\,w_{n-1,x}}{w_{n-1,y}}\right)\left\{\frac{x^{\alpha k-\lambda}\;x_{\nu+1}^{\beta_{\nu+1}k_{\nu+1}-\lambda_{\nu+1}}\cdots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\cdots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}+\lambda_{n-1}}}\right.$$

$$\left.{}_2F_1(\alpha,\beta_n;\gamma_n;x^k)\right\}$$

$$=\frac{x^{\alpha k-\lambda}\;x_{\nu+1}^{\beta_{\nu+1}k_{\nu+1}-\lambda_{\nu+1}}\cdots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\cdots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}+\lambda_{n-1}}}$$

$$F_A^{(n)}\left[\alpha,\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\;\frac{k\,x^k}{k_1\,y_1^{k_1}},\ldots,\right.$$

$$\left.\frac{k\,x^k}{k_\nu\,y_\nu^{k_\nu}},\;\frac{k\,x^k\,x_{\nu+1}^{k_{\nu+1}}}{y_{\nu+1}^{k_{\nu+1}}},\ldots,\frac{kx^k\,x_{n-1}^{k_{n-1}}}{y_{n-1}^{k_{n-1}}},\;x^k\right],$$

where for convergence

$$\sum_{i=1}^{\nu} \left| \frac{k\, x^k}{k_i\, y_i^{k_i}} \right| + \sum_{i=\nu+1}^{n-1} \left| \frac{k\, x^k\, x_i^{k_i}}{y_i^{k_i}} \right| + |x^k| < 1$$

Proof.

$$\text{L.H.S. of } (4.2.3) = \sum_{m_1,\ldots m_n=0}^{\infty} \frac{(-1)^{m_1+\ldots+m_{n-1}} (\beta_1)_{m_1} \ldots (\beta_\nu)_{m_\nu} (\alpha)_{m_n} (\beta_n)_{m_n}}{m_1! \ldots m_n! \; (\gamma_n)_{m_n}}$$

$$\frac{\omega_x^{m_1+\ldots+m_{n_1}} \; \omega_{\nu+1,x}^{m_{\nu+1}} \ldots \omega_{n-1,x}^{m_{n-1}}}{\omega_{1,y}^{m_1} \ldots \omega_{n-1,y}^{n-1}} .$$

$$\left\{ \frac{x^{\alpha k+km_n-\lambda} \; x_{\nu+1}^{\beta_{\nu+1}k_{\nu+1}-\lambda_{\nu+1}} \ldots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y_1^{(\gamma_1-1)k_1+\lambda_1} \ldots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}+\lambda_{n-1}}} \right\}$$

$$= \frac{x^{\alpha k-\lambda} \; x_{\nu+1}^{\beta_{\nu+1}\; k_{\nu+1}-\lambda_{\nu+1}} \ldots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y_1^{(\gamma_1-1)k_1+\lambda_1} \ldots y_{n-1}^{(\gamma_{n-1}-1)\; k_{n-1}+\lambda_{n-1}}}$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha)_{m_1+\ldots+m_n} (\beta_1)_{m_1} \ldots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \ldots (\gamma_n)_{m_n} \; m_1! \ldots m_n!}$$

$$\left(\frac{k\, x^k}{k_1 y_1^{k_1}}\right)^{m_1} \ldots \left(\frac{kx^k}{k_\nu y_\nu^{k_\nu}}\right)^{m_\nu} \left(\frac{kx^k \; x_{\nu+1}^{k_{\nu+1}}}{y_{\nu+1}^{K_{\nu+1}}}\right)^{m_{\nu+1}} \ldots \left(\frac{k\, x^k x_{n-1}^{k_{n-1}}}{y_{n-1}^{k_{n-1}}}\right)^{m_{n-1}} (x^k)^{m_n}$$

= R.H.S. of (4.2.3).

This completes the proof.

Similarly applying the same techniques, we obtain the following operational results :

$$(4.2.4)\quad \left(1 + \frac{w_{1,x}}{\Omega_y}\right)^{-\alpha_1} \cdots \left(1 + \frac{w_{n-1,x}}{\Omega_y}\right)^{-\alpha_{n-1}} \left\{ \frac{x_1^{k_1\beta_1-\lambda_1} \cdots x_{n-1}^{k_{n-1}\beta_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}} \; {}_2F_1(\alpha_n,\beta_n;\gamma;y^{-k}) \right\}$$

$$= \frac{x_1^{k_1\beta_1-\lambda_1} \cdots x_{n-1}^{k_{n-1}\beta_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}}$$

$$F_B^{(n)}\left[\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n;\gamma;\ \frac{k_1 x_1^{k_1}}{k\,y^k},\ldots,\frac{k_{n-1}x_{n-1}^{k_{n-1}}}{k\,y^k},\ y^{-k}\right],$$

provided that

$$|y^{-k}| < 1,\quad \left|\frac{k_i\ x_i^{k_i}}{k\ y^k}\right| < 1,\quad i = 1,\ldots,(n-1).$$

$$(4.2.5)\quad \prod_{i=1}^{n-1} \exp\left(\frac{-\Omega_x\Omega_y}{w_{i,y}}\right) \left\{ \frac{x^{\alpha k-\lambda}\ y^{\beta k-\lambda}}{y_1^{(\gamma_1-1)k_1+\lambda_1} \cdots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}+\lambda_{n-1}}} \; {}_2F_1\left(\alpha,\beta;\ \gamma_n;\ \frac{x^k y^k}{y_n^k}\right) \right\}$$

$$= \frac{x^{\alpha k-\lambda}\; y^{\beta k-\lambda}}{y_1^{(\gamma_1-1)k_1+\lambda_1} \cdots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}+\lambda_{n-1}}}$$

$$F_C^{(n)}\left[\alpha,\beta;\gamma_1,\ldots,\gamma_n;\ \frac{k^2\ x^k\ y^k}{k_1 y_1^{k_1}},\ldots,\frac{k^2\ x^k\ y^k}{k_{n-1} y_{n-1}^{k_{n-1}}},\ \frac{x^k\ y^k}{y_n^k}\right],$$

where

$$\sum_{i=1}^{n-1} \left|\left(\frac{k^2\ x^k\ y^k}{k_i y_i^{k_i}}\right)^{1/2}\right| + \left|\left(\frac{x^k\ y^k}{y_n^k}\right)^{1/2}\right| < 1$$

$$(4.2.6)\quad \left(1+\frac{\Omega_x}{\Omega_y}\right)^{-(\beta_1+\cdots+\beta_\nu)} \exp\left[-\frac{\Omega_x}{\Omega_y}(\Omega_{\nu+1,x}+\cdots+\Omega_{n-1,x})\right]$$

$$\left\{\frac{x^{\alpha k-\lambda}\ x_{\nu+1}^{\beta_{\nu+1}\ k_{\nu+1}-\lambda_{\nu+1}} \cdots x_{n-1}^{\beta_{n-1}\ k_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}}\ {}_2F_1\left(\alpha,\beta_n;\gamma;\frac{x^k}{y^k}\right)\right\}$$

$$= \frac{x^{\alpha k-\lambda}\ x_{\nu+1}^{\beta_{\nu+1}\ k_{\nu+1}-\lambda_{\nu+1}} \cdots x_{n-1}^{\beta_{n-1}\ k_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}}$$

$$F_D^{(n)}\left[\alpha,\beta_1,\ldots,\beta_n;\gamma;\ \frac{x^k}{y^k},\ldots,\frac{x^k}{y^k},\ \frac{k_{\nu+1}x^k x_{\nu+1}^{k_{\nu+1}}}{y^k},\right.$$

$$\left.\ldots,\ \frac{k_{n-1}\ x^k\ x_{n-1}^{k_{n-1}}}{y^k},\ \frac{x^k}{y^k}\right],$$

provided that

$$\nu\left|\frac{x^k}{y^k}\right| + \left|\frac{k_{\nu+1}\ x^k\ x^{k_{\nu+1}}}{y^k}\right| + \cdots + \left|\frac{k_{n-1}\ x^k\ x_{n-1}^{k_{n-1}}}{y^k}\right| < 1$$

$$(4.2.7) \quad \exp\left[-\Omega_x\left\{\frac{1}{w_{1,y}}+\dots+\frac{1}{w_{n-1,y}}\right\}\right]$$

$$\left\{\frac{x^{\alpha k-\lambda}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\cdots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}+\lambda_{n-1}}}\ {}_1F_1(\alpha;\ \gamma_n;x^k)\right\}$$

$$=\frac{x^{\alpha k-\lambda}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\cdots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}+\lambda_{n-1}}}$$

$$\Psi_2^{(n)}(\alpha;\gamma_1,\dots,\ \gamma_n;\ \frac{kx^k}{k_1y_1^{k_1}},\dots,\ \frac{kx^k}{k_{n-1}y_{n-1}^{k_{n-1}}},\ x^k),$$

$$(4.2.8) \quad \exp\left[-\left(\frac{w_{1,x}+\dots+w_{n-1,x}}{\Omega_y}\right)\right]\ \left\{\frac{x_1^{\beta_1k_1-\lambda_1}\cdots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}}\right.$$

$$\left.{}_1F_1(\beta_n,\gamma\ ;\ \frac{1}{y^k})\right\}$$

$$=\frac{x_1^{\beta_1k_1-\lambda_1}\ \cdots\ x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}}$$

$$\Phi_2^{(n)}\ (\beta_1,\dots,\beta_n;\gamma\ ;\ \frac{k_1\ x_1^{k_1}}{ky^k},\dots,\ \frac{k_{n-1}\ x_{n-1}^{k_{n-1}}}{ky^k},\ \frac{1}{y^k})$$

$$(4.2.9) \quad \left(1+\frac{\Omega_x}{\Omega_y}\right)^{-(\beta_1+\dots+\beta_\nu)}\ \exp\left[-\frac{\Omega_x}{w_{i,y}}(w_{\nu+1,x}+\dots+w_{n-1,x})\right]$$

$$\left\{\frac{x^{\alpha k-\lambda}\, x_{\nu+1}^{\beta_{\nu+1}k_{\nu+1}-\lambda_{\nu+1}} \cdots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}\, y_i^{(\gamma_i-1)k_i+\lambda_i}}\; {}_2F_1(\alpha,\beta_n;\gamma_i;\frac{x^k}{y_i^{k_i}})\right\}$$

$$= \frac{x^{\alpha k-\lambda}\, x_{\nu+1}^{\beta_{\nu+1}k_{\nu+1}-\lambda_{\nu+1}} \cdots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}\, y_i^{(\gamma_i-1)k_i+\lambda_i}}$$

$$\begin{smallmatrix}(2)\\(1)\end{smallmatrix}E_D^{(n)}\left[\alpha,\beta_1,\ldots,\beta_n;\gamma,\gamma_i;\frac{x^k}{y^k},\ldots,\frac{x^k}{y^k},\frac{k\,k_{\nu+1}x^k\, x_{\nu+1}^{k_{\nu+1}}}{k_i\, y_i^{k_i}},\ldots,\frac{k\,k_{n-1}\,x^k\, x_{n-1}^{k_{n-1}}}{k_i\, y_i^{k_i}},\frac{x^k}{y_i^{k_i}}\right]$$

where i is any positive integer

$$(4.2.10)\quad (1+\frac{w_{i,y}}{\Omega_y})^{-(\beta_1+\ldots+\beta_\nu)}$$

$$\exp\left[-\frac{\Omega_x}{\Omega_y}\{w_{\nu+1,x}+\ldots+w_{n-1,x}\}\right]$$

$$\left\{\frac{y_i^{\alpha_i k_i-\lambda_i}\, x^{\alpha k-\lambda}\, x_{\nu+1}^{\beta_{\nu+1}k_{\nu+1}-\lambda_{\nu+1}} \cdots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}}\right.$$

$$\left. {}_2F_1(\alpha,\beta_n;\gamma;\frac{x^k}{y^k})\right\}$$

$$= \frac{y_i^{\alpha_i k_i-\lambda_i}\, x^{\alpha k-\lambda}\, x^{\beta_{\nu+1}k_{\nu+1}-\lambda_{\nu+1}} \cdots x_{n-1}^{\beta_{n-1}k_{n-1}-\lambda_{n-1}}}{y^{(\gamma-1)k+\lambda}}$$

$$ {}^{(\nu)}_{(2)}E_D^{(n)}\left[\alpha_i,\alpha,\beta_1,\ldots,\beta_n;\gamma;\frac{k_i y_i^{k_i}}{k\,y^k},\ldots,\frac{k_i y_i^{k_i}}{k\,y^k},\frac{k_{\nu+1}x^k x_{\nu+1}^{k_{\nu+1}}}{y^k},\ldots,\frac{k_{n-1}\,x^k\,x_{n-1}^{k_{n-1}}}{y^k},\frac{x^k}{y^k}\right] $$

where i is any positive integer.

and

$$ (4.2.11)\quad \exp\left[-\Omega_x\Omega_y\left(\frac{1}{w_{1,y}}+\ldots+\frac{1}{w_{\nu,y}}\right)\right] $$

$$ \exp\left[-w_{i,x}\,\Omega_y\left(\frac{1}{w_{\nu+1,y}}+\ldots+\frac{1}{w_{n-1,y}}\right)\right] $$

$$ \left\{\frac{x^{\alpha k-\lambda}\;y^{\beta k-\lambda}\;x_i^{k_i\alpha_i-\lambda_i}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\ldots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}+\lambda_{n-1}}}\;{}_2F_1(\alpha_i,\beta;\gamma_n;x_i^{k_i}y^k)\right\} $$

$$ =\frac{x^{\alpha k-\lambda}\;y^{\beta k-\lambda}\;x_i^{\alpha_i k_i-\lambda_i}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\ldots y_{n-1}^{(\gamma_{n-1}-1)k_{n-1}}} $$

$$ {}^{(\nu)}_{(1)}E_C^{(n)}\left[\alpha,\alpha_i,\beta;\gamma_1,\ldots,\gamma_n;\frac{k^2\;x^k\;y^k}{k_1\;y_1^{k_1}},\ldots,\frac{k^2x^ky^k}{k_\nu\;y_\nu^{k_\nu}},\frac{k_i k x_i^{k_i}y^k}{k_{\nu+1}y_{\nu+1}^{k_{\nu+1}}},\ldots,\frac{k_i\;k\;x_i^{k_i}\;y^k}{k_{n-1}y_{n-1}^{k_{n-1}}},x_i^{k_i}y^k\right] $$

where i is any positive integer.

4.3 Some Particular Cases. In this section, we shall give some special cases of the results obtained in Section §4.2.

For $n = 3$, $\nu = 1$ (4.2.9) gives

$$(4.3.1)\quad \left(1 + \frac{\Omega_x}{\Omega_y}\right)^{-\beta_1} \exp\left(\frac{-\Omega_x w_{2,x}}{w_{i,y}}\right)$$

$$\left\{\frac{x^{\alpha k-\lambda}\, x_2^{\beta_2 k_2-\lambda_2}}{y^{(\gamma-1)k+\lambda}\, y_i^{(\gamma_i-1)k_i+\lambda_i}}\ {}_2F_1\left(\alpha, \beta_3; \gamma_i; \frac{x^k}{y^k}\right)\right\}$$

$$= \frac{x^{\alpha k-\lambda}\, x_2^{\beta_2 k_2-\lambda_2}}{y^{(\gamma-1)k+\lambda}\, y_i^{(\gamma_i-1)k_i+\lambda_i}}$$

$$F_G\left(\alpha, \beta_1, \beta_2, \beta_3; \gamma, \gamma_i; \frac{x^k}{y^k}, \frac{k\, k_2\, x^k\, x_2^{k_2}}{k_i y_i^{k_i}}, \frac{x^k}{y_i^{k_i}}\right)$$

and for $n = 4$, $\nu = 3$, (4.2.9) gives

$$(4.3.2)\quad \left(1 + \frac{\Omega_x}{\Omega_y}\right)^{-(\beta_1+\beta_2+\beta_3)} \left\{\frac{x^{\alpha k-\lambda}}{y^{(\gamma-1)k+\lambda}}\ {}_2F_1(\alpha, \beta_4; \gamma_i, x^k)\right\}$$

$$= \frac{x^{\alpha k-\lambda}}{y^{(\gamma-1)k+\lambda}}$$

$$K_{11}\left(\alpha, \alpha, \alpha, \alpha; \beta_1, \beta_2, \beta_3, \beta_4; \gamma, \gamma, \gamma, \gamma_i; \frac{x^k}{y^k}, \frac{x^k}{y^k}, \frac{x^k}{y^k}, x^k\right)$$

while for $n = 4$, $\nu = 2$, (4.2.9) gives

$$(4.3.3) \quad (1 + \frac{\Omega_x}{\Omega_y})^{-(\beta_1+\beta_2)} \exp(- \frac{\Omega_x\ w_{3,x}}{w_{i,y}})$$

$$\{\frac{x^{\alpha k-\lambda}\ x_3^{\beta_3 k_3-\lambda_3}}{y^{(\gamma-1)k+}\ y_i^{(\gamma_i-1)k_i+\lambda_i}}\ {}_2F_1(\alpha,\beta_4;\gamma_i;\ \frac{x^k}{y_i^{k_i}})\}$$

$$= \frac{x^{\alpha k-\lambda}\ x_3^{\beta_3 k_3-\lambda_3}}{y^{(\gamma-1)k+\lambda}\ y_i^{(\gamma_i-1)k_i+\lambda_i}}$$

$$K_{12}(\alpha,\alpha,\alpha,\alpha;\beta_1,\beta_2,\beta_3,\beta_4;\gamma,\gamma,\gamma_i,\gamma_i;\ \frac{x^k}{y^k},\ \frac{x^k}{y^k},\ \frac{kk_3x^k x_3^{k_3}}{k_i\ y_i^{k_i}},\ \frac{x^k}{y_i^{k_i}})$$

Similarly for $n = 3$, $\nu = 1$ (4.2.10) gives

$$(4.3.4) \quad (1 + \frac{w_{i,y}}{\Omega_y})^{-\beta_1} \exp(- \frac{\Omega_x w_{2,x}}{\Omega_y})$$

$$\{\frac{y_i^{\alpha_i k_i-\lambda_i}\ x^{\alpha k-\lambda}\ x_2^{\beta_2 k_2-\lambda_2}}{y^{(\gamma-1)k+\lambda}}\ {}_2F_1(\alpha,\beta_3;\gamma;\ \frac{x^k}{y^k})\}$$

$$= \frac{x^{\alpha k-\lambda}\ y_i^{\alpha_i k_i-\lambda_i}\ x_2^{\beta_2 k_2-\lambda_2}}{y^{(\gamma-1)k+\lambda}}$$

$$F_S(\alpha_i,\alpha,\alpha,\beta_1,\beta_2,\beta_3;\gamma,\gamma,\gamma;\ \frac{k_i\ y_i^{k_i}}{k\ y^k},\ \frac{k_2\ x^k\ x_2^{k_2}}{y^k},\ \frac{x^k}{y^k})$$

and for $n = 4$, $\nu = 3$ (4.2.10) gives

$$(4.3.5)\quad \left(1+\frac{w_{i,y}}{\Omega_y}\right)^{-(\beta_1+\beta_2+\beta_3)}\left\{\frac{y_i^{\alpha_i k_i-\lambda_i}\,x^{\alpha k-\lambda}}{y^{(\gamma-1)k+\lambda}}\;{}_2F_1\left(\alpha,\beta_4;\gamma;\frac{x^k}{y^k}\right)\right\}$$

$$=\frac{x^{\alpha k-\lambda}\,y_i^{\alpha_i k_i-\lambda_i}}{y^{(\gamma-1)k+\lambda}}$$

$$K_{15}\left(\alpha_i,\alpha_i,\alpha_i,\alpha;\beta_1,\beta_2,\beta_3,\beta_4;\gamma,\gamma,\gamma,\gamma;\ \frac{k_i y_i^{k_i}}{ky^k},\ \frac{k_i y_i^{k_i}}{ky^k},\ \frac{k_i y_i^{k_i}}{ky^k},\ \frac{x^k}{y^k}\right),$$

while for $n = 4$, $\nu = 2$, (4.2.10) gives

$$(4.3.6)\quad \left(1+\frac{w_{i,y}}{\Omega_y}\right)^{-(\beta_1+\beta_2)}\exp\left(-\frac{\Omega_x w_{3,x}}{\Omega_y}\right)$$

$$\left\{\frac{y_i^{\alpha_i k_i-\lambda_i}\,x^{\alpha k-\lambda}\,x_3^{\beta_3 k_3-\lambda_3}}{y^{(\gamma-1)k+\lambda}}\;{}_2F_1\left(\alpha,\beta_4,\gamma;\frac{x^k}{y^k}\right)\right\}$$

$$=\frac{x^{\alpha k-\lambda}\,y_i^{\alpha_i k_i-\lambda_i}\,x_3^{\beta_3 k_3-\lambda_3}}{y^{(\gamma-1)k+\lambda}}$$

$$K_{20}\left(\alpha_i,\alpha_i,\beta_3,\beta_4;\beta_1,\beta_2,\alpha,\alpha;\gamma,\gamma,\gamma,\gamma;\ \frac{k_i y_i^{k_i}}{ky^k},\frac{k_i y_i^{k_i}}{ky^k},\ \frac{k_3 x^k x_3^{k_3}}{y^k},\ \frac{x^k}{y^k}\right)$$

For $n = 2$, $\nu = 1$, (4.2.11) gives

$$(4.3.7)\quad \exp\left(-\frac{\Omega_x\Omega_y}{w_{1,y}}\right)\left\{\frac{x^{\alpha k-\lambda}\,y^{\beta k-\lambda}}{y_1^{(\gamma_1-1)k_1+\lambda}}\;{}_2F_1(\alpha_i,\beta;\gamma_2;y^k)\right\}$$

$$=\frac{x^{\alpha k-\lambda}\,y^{\beta k-\lambda}}{y_1^{(\gamma_1-1)k_1+\lambda}}\quad F_2\left(\beta_2,\alpha,\alpha_i,\gamma_1,\gamma_2;\ \frac{k^2\,x^k\,y^k}{k_1 y_1},\ y^k\right)$$

and for $n = 3$, $\nu = 1$, (4.2.11) gives

$$(4.3.8)\quad \exp\left(-\frac{\Omega_x\Omega_y}{w_{1,y}}\right)\exp\left(-\frac{w_{i,x}\,\Omega_y}{w_{2,y}}\right)$$

$$\left\{\frac{x^{\alpha k-\lambda}\;y^{\beta k-\lambda}\;x_i^{\alpha_i k_i-\lambda_i}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\;y_2^{(\gamma_2-1)k_2+\lambda_2}}\;{}_2F_1(\alpha_i,\beta_i,\gamma_3;x_i^{k_i}y^k)\right\}$$

$$=\frac{x^{\alpha k-\lambda}\;y^{\beta k-\lambda}\;x_i^{\alpha_i k_i-\lambda_i}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\;y_2^{(\gamma_2-1)k_2+\lambda_2}}$$

$$F_E(\beta,\beta,\beta,\alpha,\alpha_i,\alpha_i;\gamma_1,\gamma_2,\gamma_3;\frac{k^2x^ky^k}{k_1y_1^{k_1}},\;\frac{k_ik\;x_i^{k_i}y^k}{k_2\;y_2^{k_2}},\;x_i^{k_i}\;y^k)$$

Again for $n = 4$, $\nu = 3$, (4.2.11) gives

$$(4.3.9)\quad \exp\left[-\Omega_x\Omega_y\left(\frac{1}{w_{1,y}}+\frac{1}{w_{2,y}}+\frac{1}{w_{3,y}}\right)\right]$$

$$\left\{\frac{x^{\alpha k-\lambda}\;y^{\beta k-\lambda}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\;y_2^{(\gamma_2-1)k_2+\lambda_2}\;y_3^{(\gamma_3-1)k_3+\lambda_3}}\;{}_2F_1(\alpha_i,\beta,\gamma_4;y^k)\right\}$$

$$=\frac{x^{\alpha k-\lambda}\;y^{\beta k-\lambda}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\;y_2^{(\gamma_2-1)k_2+\lambda_2}\;y_3^{(\gamma_3-1)k_3+\lambda_3}}$$

$$K_2(\beta,\beta,\beta,\beta;\alpha,\alpha,\alpha,\alpha_i;\gamma_1,\gamma_2,\gamma_3,\gamma_4;\frac{k^2x^ky^k}{k_1y_1^{k_1}},\;\frac{k^2x^ky^k}{k_2y_2^{k_2}},\;\frac{k^2x^ky^k}{k_3y_3^{k_3}},\;y^k)$$

While for $n = 4$, $\nu = 2$, (4.2.11) gives

$$(4.3.10)\quad \exp\left(-\frac{\Omega_x\Omega_y}{w_{1,y}}\right)\exp\left(-\frac{\Omega_x\Omega_y}{w_{2,y}}\right)\exp\left(-\frac{\Omega_y\, w_{i,x}}{w_{3,y}}\right)$$

$$\left\{\frac{x^{\alpha k-\lambda}\, y^{\beta k-\lambda}\, x_i^{\alpha_i k_i-\lambda_i}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\, y_2^{(\gamma_2-1)k_2+\lambda_2}\, y_3^{(\gamma_3-1)k_3+\lambda_3}}\; {}_2F_1(\alpha_i,\beta;\gamma_4;x_i^{k_i}y^k)\right\}$$

$$= \frac{x^{\alpha k-\lambda}\, y^{\beta k-\lambda}\, x_i^{\alpha_i k_i-\lambda_i}}{y_1^{(\gamma_1-1)k_1+\lambda_1}\, y_2^{(\gamma_2-1)k_2+\lambda_2}\, y_3^{(\gamma_3-1)k_3+\lambda_3}}$$

$$K_5\left(\beta,\beta,\beta,\beta;\alpha,\alpha,\alpha_i,\alpha_i;\gamma_1,\gamma_2,\gamma_3,\gamma_4;\ \frac{k^2x^ky^k}{k_1y_1^{k_1}},\ \frac{k^2x^ky^k}{k_2y_2^{k_2}},\right.$$

$$\left.\frac{k_ik\,x_i^{k_i}y^k}{k_3y_3^{k_3}},\ x_i^{k_i}\,y^k\right),$$

where $K_2$, $K_5$, $K_{11}$, $K_{12}$, $K_{15}$ and $K_{20}$ are hypergeometric functions of four variables introduced by Exton [5].

Similarly several other particular cases can also be obtained.

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CHAPTER V

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# NEW MULTIDIMENSIONAL WHITTAKER TRANSFORM OF MULTIPLE HYPERGEOMETRIC FUNCTIONS IN SEVERAL VARIABLES

5.1 Introduction. Recently Chandel and Dwivedi [2,3], introduced and studied the generalized Whittaker transform

$$(5.1.1)\quad \Omega^{\lambda,\mu,\nu}_{\alpha_1,\ldots,\alpha_r;\sigma}\{\ \}$$

$$= \frac{\lambda^{\sigma+\alpha_1+\ldots+\alpha_r}\ \Gamma(1+\sigma-\mu+\alpha_1+\ldots+\alpha_r)\ \Gamma(\alpha_1+\ldots+\alpha_r)}{\Gamma(\sigma+\frac{1}{2}+\alpha_1+\ldots+\alpha_r\pm\nu)\ \prod_{i=1}^{r}\Gamma(\alpha_i)}$$

$$\int_0^\infty\cdots\int_0^\infty x_1^{\alpha_1-1}\cdots x_r^{\alpha_r-1}(x_1+\ldots+x_r)^{\sigma}\ e^{-\frac{\lambda}{2}(x_1+\ldots+x_r)}\ W_{\mu,\nu}[\lambda(x_1+\ldots+x_r)]\{\ \}dx_1\ldots dx_r$$

where $Re(\lambda)>0$, $Re(\sigma+\frac{1}{2}+\alpha_1+\ldots+\alpha_r\pm\nu)>0$ and $Re(\alpha_j)>0$, $j=1,\ldots,r$, to evaluate certain multiple integrals involving Exton [6], Lauricella [7] and Srivastava and Daoust [8] functions of several variables. Further, Chandel and Dwivedi [4] have also studied a more generalized Whittaker transform

(5.1.2) $$T^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{\ \}$$

$$= \frac{K\,\lambda^{\sigma+\beta_1+\ldots+\beta_r}\,\Gamma(\sigma+1-\mu+\beta_1+\ldots+\beta_r)\,\Gamma(\beta_1+\ldots+\beta_r)}{\Gamma(\sigma+\frac{1}{2}\pm\nu+\ldots+\beta_1+\ldots+\beta_r)\prod_{i=1}^{r}\Gamma(\beta_i)}$$

$$\int_0^\infty\ldots\int_0^\infty \exp\left[-\frac{\lambda}{2}\left(\sum_{j=1}^{r}(\alpha_1^j x_1+\ldots+\alpha_r^j x_r\right)\right]$$

$$\prod_{j=1}^{r}(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)^{\beta_j-1}\left[\sum_{j=1}^{r}(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)\right]^{\sigma}$$

$$W_{\mu,\nu}\left[\lambda\sum_{j=1}^{r}(\alpha_1^j x_1+\ldots+\alpha_r^j\ x_r)\right]\{\ \}\,dx_1\ldots dx_r$$

where $\mathrm{Re}\,(\lambda) > 0$, $\mathrm{Re}\,(\sigma+\frac{1}{2}+\beta_1+\ldots+\beta_r\pm\nu) > 0$, $\mathrm{Re}\,(\sigma+1-\mu+\beta_1+\ldots+\beta_r) > 0$, $\mathrm{Re}\,(\beta_j) > 0$, $j = 1,\ldots,r$ and

$$K = \begin{vmatrix} \alpha_1^1 & \alpha_2^1 & \ldots & \alpha_r^1 \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_r^2 \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^r & \alpha_2^r & \ldots & \alpha_r^r \end{vmatrix} \neq 0$$

In this chapter, we introduce the following new multidimensional Whittaker transform

(5.1.3) $$S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{\ \} =$$

$$\frac{2K\,\lambda^{\beta_1+\ldots+\beta_r+\sigma}\,\Gamma(\beta_1+\ldots+\beta_r)\,\Gamma[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu]}{\Gamma(\beta_1)\ldots\Gamma(\beta_r)\,\Gamma(1+\beta_1+\ldots+\beta_r+\sigma)\Gamma(\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\pm\nu)}$$

$$\int_0^\infty\ldots\int_0^\infty(\alpha_1' x_1+\ldots+\alpha_r' x_r)^{\beta_1-1}\ldots(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^{\beta_r-1}$$

$$[\sum_{j=1}^r(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)]^\sigma\, W_{\mu,\nu}[\lambda\sum_{j=1}^r(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)]$$

$$W_{-\mu,\nu}[\lambda\sum_{j=1}^r(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)]\,\{\quad\}\,dx_1\ldots dx_r,$$

where $\mathrm{Re}(\beta_1+\ldots+\beta_r+\sigma)\geqslant 2\,\mathrm{Re}\,|\nu|-1$ and

$$K=\begin{vmatrix}\alpha_1' & \alpha_2' & \ldots & \alpha_r' \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_r^2 \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^r & \alpha_2^r & \ldots & \alpha_r^r\end{vmatrix}\neq 0,$$

to evaluate certain new multiple integrals involving multiple hypergeometric functions of Srivastava and Daoust [8]. We shall also discuss some of their interesting special cases, to obtain multi dimensional Whittaker transforms of hypergeometric functions of several variables defined by Exton [6], Lauricella [7] and Chandel [1].

5.2 <u>Some Properties of the Operator</u>. Here we write the following remarkable properties of the operator:

$$(5.2.1)\quad S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{\;\} \equiv S^{\lambda,-\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{\;\} \equiv S^{\lambda,-\mu,-\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{\;\}$$

$$\equiv S^{\lambda,\mu,-\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{\;\}$$

$$(5.2.2)\quad S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{1\} = 1.$$

5.3 Whittaker transforms of generalized nature. First of all we establish

$$(5.3.1)\quad S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{(\alpha_1' x_1+\ldots+\alpha_r' x_t)^{m_1\xi_1}\ldots(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^{m_r\xi_r}\}$$

$$= \frac{(\beta_1)_{m_1\xi_1}\ldots(\beta_r)_{m_r\xi_r}(1+\beta_1+\ldots+\beta_r+\sigma)_{m_1\xi_1+\ldots+m_r\xi_r}\left(\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\pm\nu\right)_{\left(\frac{m_1\xi_1+\ldots+m_r\xi_r}{2}\right)}}{\lambda^{m_1\xi_1+\ldots+m_r\xi_r}(\beta_1+\ldots+\beta_r)_{m_1\xi_1+\ldots+m_r\xi_r}\left(1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu\right)_{\frac{m_1\xi_1+\ldots+m_r\xi_r}{2}}}$$

An appeal to the above result gives the following results involving Srivastava and Daoust function [8]

$$(5.3.2)\quad S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\left\{F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left[\begin{array}{l}[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\Psi',\ldots,\Psi^{(r)}]:\end{array}\right.\right.$$

$[(b'):\Phi'];\ldots;[(b^{(r)}):\Phi^{(r)}];$

$[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];$

$$u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)^{\xi_1},\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^{\xi_r}\Big]\ \}$$

$$= F^{A+3:B'+1;\ldots;B^{(r)}+1}_{C+3:D';\ldots;D^{(r)}}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}],[1+\beta_1+\ldots+\beta_r+\sigma:\xi_1,\ldots,\xi_r],\\ [(c):\Psi',\ldots,\Psi^{(r)}],[\beta_1+\ldots+\beta_r:\xi_1,\ldots,\xi_r],\end{matrix}\right.$$

$[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\pm\nu:\frac{\xi_1}{2},\ldots,\frac{\xi_r}{2}]:[(b'):\Phi']\ ,\ [\beta_1:\xi_1];\ldots;[(b^{(r)}):$

$[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu:\frac{\xi_1}{2},\ldots,\frac{\xi_r}{2}]:[(d'):\delta'];\ldots;[(d^{(r)}):$

$$\left.\begin{matrix}\Phi^{(r)}],[\beta_r:\xi_r];\\ \delta^{(r)}];\end{matrix}\quad u_1\lambda^{-\xi_1},\ldots,u_r\lambda^{-\xi_r}\right],$$

provided that $\mathrm{Re}(\beta_1+\ldots+\beta_r+\sigma) > 2\ \mathrm{Re}\ |\nu| - 1$, $K \neq 0$ and

$$1-\xi_1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{i}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)}>0, i=1,\ldots,r$$

(5.3.3) $S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\ \{F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\Big[$

$[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\Phi']\ ;\ldots;\ [(b^{(r)}):\Phi^{(r)}]\ ;$

$[(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta']\ ;\ldots;\ [(d^{(r)}):\delta^{(r)}]\ ;$

$$u_1(\alpha_1' x_1+\dots+\alpha_r' x_r)^{\xi_1}\Big[\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\Big]^{\eta_1},\dots$$

$$,\, u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)^{\xi_r}\Big[\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\Big]^{\eta_r}\Bigg]\Big\}$$

$$= F^{A+3:B'+1;\dots;B^{(r)}+1}_{C+3:D'\dots;D^{(r)}}\Bigg[\begin{array}{l}[(a):\theta',\dots,\theta^{(r)}],\,[1+\beta_1+\dots+\beta_r+\sigma : \\ [(c):\Psi',\dots,\Psi^{(r)}],\,[\beta_1+\dots+\beta_r :\end{array}$$

$$\xi_1+\eta_1,\dots,\xi_r+\eta_r],\ \Big[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}\pm\nu : \frac{\xi_1+\eta_1}{2},\dots,\frac{\xi_r+\eta_r}{2}\Big]:$$

$$\xi_1,\dots,\xi_r],\ \Big[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu : \frac{\xi_1+\eta_1}{2},\dots,\frac{\xi_r+\eta_r}{2}\Big]:$$

$$[(b'):\Phi'],\ [\beta_1:\xi_1],\dots,[(b^{(r)}):\Phi^{(r)}],\ [\beta_r:\xi_r];$$

$$[(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];$$

$$u_1\,\lambda^{-(\xi_1+\eta_1)},\dots,u_r^{\quad -(\xi_r+\eta_r)}\Bigg],$$

where $\mathrm{Re}(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$, $K \neq 0$ and

$$1-\xi_i-\eta_i+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)} > 0,\ i = 1,\dots,r.$$

5.4. Special Cases. For particular interest, specialising the parameters here in this section, we obtain the following results :

$$(5.4.1)\quad S^{\lambda,\frac{1}{4},\frac{1}{4}}_{\beta_1,\ldots,\beta_r;-\frac{1}{2}}\{\exp[\sum_{j=1}^{r} u_j(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)]\}$$

$$= F_D^{(r)}[\beta_1+\ldots+\beta_r+\tfrac{1}{2},\beta_1,\ldots,\beta_r;\beta_1+\ldots+\beta_r+1;\frac{u_1}{\lambda},\ldots,\frac{u_r}{\lambda}],$$

provided that Re $(\beta_1+\ldots+\beta_r) > 0$, $K \neq 0$ and $|\frac{u_1}{\lambda}| < 1,\ldots,|\frac{u_r}{\lambda}| < 1$.

$$(5.4.2)\quad S^{\lambda,\frac{1}{4},\frac{1}{4}}_{\beta_1,\ldots,\beta_r;-\frac{1}{2}}\{\Psi_2^{(r)}(\beta_1+\ldots+\beta_r+1;\gamma_1,\ldots,\gamma_r;$$

$$u_1(\alpha_1' x_1+\ldots+\alpha_r' x_r),\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^\alpha x_r))\}$$

$$= F_A^{(r)}(\beta_1+\ldots+\beta_r+\tfrac{1}{2},\beta_1,\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;\frac{u_1}{\lambda},\ldots,\frac{u_r}{\lambda}),$$

where $\mathrm{Re}(\beta_1+\ldots+\beta_r) > 0$, $K \neq 0$ and $|\frac{u_1}{\lambda}| + \ldots + |\frac{u_r}{\lambda}| < 1$.

$$(5.4.3)\quad S^{\lambda,\frac{1}{4},\frac{1}{4}}_{\beta_1,\ldots,\beta_r;\sigma}\{\Psi_\lambda^{(r)}(C,\beta_1,\ldots,\beta_r;u_1(\alpha_1' x_1+\ldots+\alpha_r' x_r),\ldots$$

$$,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r))\}$$

$$= {}_3F_2\left[\begin{matrix} C,1+\beta_1+\ldots+\beta_r+\sigma,\beta_1+\ldots+\beta_r+\sigma+\frac{1}{2}; \\ \beta_1+\ldots+\beta_r,\beta_1+\ldots+\beta_r+\sigma+\frac{3}{2}; \end{matrix}\ \frac{u_1+\ldots+u_r}{\lambda}\right]$$

valid if $\mathrm{Re}(\beta_1+\ldots+\beta_r+\sigma) > -\frac{1}{2}$ and $K \neq 0$.

$$(5.4.4)\quad S^{\lambda,\frac{1}{4},\frac{1}{4}}_{\beta_1,\dots,\beta_r;\sigma}\{F_C^{(r)}(a,b;\beta_1,\dots,\beta_r;u_1(\alpha_1'x_1+\dots+\alpha_r'x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r))\}$$

$$= {}_4F_2\left[\begin{matrix} a,b,1+\beta_1+\dots+\beta_r+\sigma,\frac{1}{2}+\beta_1+\dots+\beta_r+\sigma; \\ \beta_1+\dots+\beta_r,\beta_1+\dots+\beta_r+\sigma+\frac{3}{2}; \end{matrix}\ \frac{u_1+\dots+u_r}{\lambda}\right]$$

where $\mathrm{Re}(\beta_1+\dots+\beta_r+\sigma) > -\frac{1}{2}$, $K \neq 0$, and a or b is any negative integer.

$$(5.4.5)\quad S^{\lambda,\frac{1}{4},\frac{1}{4}}_{\beta_1,\dots,\beta_r;-\frac{1}{2}}\{\Psi_2^{(r)}(\gamma;\beta_1,\dots,\beta_r;u_1(\alpha_1'x_1+\dots+\alpha_r'x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r))\}$$

$$= {}_2F_1\left[\begin{matrix} \frac{1}{2}+\beta_1+\dots+\beta_r,\gamma; \\ \beta_1+\dots+\beta_r+1; \end{matrix}\ \frac{u_1+\dots+u_r}{\lambda}\right],$$

where $\mathrm{Re}(\beta_1+\dots+\beta_r) > 0$ and $K \neq 0$

$$(5.4.6)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\{F^{3:-;\dots;-}_{2:1;\dots;1}\Big[$$

$$[\beta_1+\dots+\beta_r:1,\dots,1],[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma\pm\mu:\tfrac{1}{2},\dots,\tfrac{1}{2})]:$$

$$[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}\pm\nu:\frac{1}{2},\dots,\frac{1}{2}]:$$

$$-;\ldots;-;$$

$$\left.\left.[d_1:1]\ ;\ldots;\ [d_r:1]\ ;\ u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r),\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)\right]\right\}$$

$$= F_A^{(r)}(1+\beta_1+\ldots+\beta_r+\sigma,\beta_1,\ldots,\beta_r;d_1,\ldots,d_r;\ \frac{u_1}{\lambda}\ ,\ldots,\ \frac{u_r}{\lambda}),$$

provided that $|\frac{u_1}{\lambda}|\ +\ldots+\ |\frac{u_r}{\lambda}| < 1$, $\mathrm{Re}(\beta_1+\ldots+\beta_r+\sigma) > 2\ \mathrm{Re}\ |\nu|\ -1$ and $K \neq 0$

$$(5.4.7)\quad S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\ \{F^{2:1;\ldots;1}_{3:-;\ldots;-}\left[\begin{matrix}[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\ \pm\mu: \\ [1+\beta_1+\ldots+\beta_r+\sigma:1,\ldots,1]\ ,\end{matrix}\right.$$

$$\frac{1}{2},\ldots,\frac{1}{2}]\ :[\gamma_1:1]\ ;\ldots;\ [\gamma_r:1]\ ;$$

$$[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\ \pm\nu\ :\ \frac{1}{2},\ldots,\frac{1}{2}\ ]:\ -;\ldots;-;$$

$$\left.\left.u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r),\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)\right]\right\}$$

$$= F_B^{(r)}\ [\ \beta_1,\ldots,\beta_r,\ \gamma_1,\ldots,\gamma_r;\beta_1+\ldots+\beta_r\ ;\ \frac{u_1}{\lambda}\ ,\ldots,\ \frac{u_r}{\lambda}\ ],$$

valid if $\mathrm{Re}(\beta_1+\ldots+\beta_r+\sigma) > 2\ \mathrm{Re}\ |\nu|\ -\ 1$, $K \neq 0$ and all $|\frac{u_i}{\lambda}| < 1$, $i = 1,\ldots,r$.

$$(5.4.8)\quad S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\ \{F^{4:-;\ldots;-}_{2:2;\ldots;2}\left[\begin{matrix}[\beta_1+\ldots+\beta_r:1,\ldots,1], \\ [\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\ \pm\nu\ :\end{matrix}\right.$$

$$[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma) \pm\mu : \tfrac{1}{2},\dots,\tfrac{1}{2}],[\delta:1,\dots,1] :-;\dots;-;$$

$$\tfrac{1}{2},\dots,\tfrac{1}{2}] : [\beta_1:1] , [\gamma_1:1] ;\dots; [\beta_r:1] ,[\gamma_r:1] ;$$

$$\left. u_1(\alpha_1' x_1+\dots+\alpha_r' x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r) \right] \}$$

$$= F_C^{(r)} [1+\beta_1+\dots+\beta_r+\sigma,\delta;\gamma_1,\dots,\gamma_r ; \frac{u_1}{\lambda} ,\dots, \frac{u_r}{\lambda}] ,$$

provided that $\sum_{i=1}^{r} |(\frac{u_i}{\lambda})^{\frac{1}{2}}| < 1$, $\mathrm{Re}(\beta_1+\dots+\beta_r+\sigma) > 2\ \mathrm{Re}\ |\nu| - 1$ and $K \neq 0$

$$(5.4.9) \quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma} \{F^{2:-;\dots;-}_{2;-;\dots;-} \left[ \begin{matrix} [1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma) \pm\mu : \\ [\frac{1+\beta_1+\dots+\beta_r+\sigma}{2} \pm\nu : \end{matrix} \right.$$

$$\left. \begin{matrix} \frac{1}{2},\dots,\frac{1}{2}] :-;\dots;-; \\ \frac{1}{2},\dots,\frac{1}{2}] :-;\dots;-; \end{matrix} \quad u_1(\alpha_1' x_1+\dots+\alpha_r' x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r) \right] \}$$

$$= F_D^{(r)} (1+\beta_1+\dots+\beta_r+\sigma;\beta_1,\dots,\beta_r;\beta_1+\dots+\beta_r; \frac{u_1}{\lambda} ,\dots, \frac{u_r}{\lambda}) ,$$

where all the conditions of (5.4.7) are satisfied

$$(5.4.10) \quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma} \{F^{2:-;\dots;-}_{3:-;\dots;-} \left[ \begin{matrix} [1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma) \pm\mu : \\ [\frac{1+\beta_1+\dots+\beta_r+\sigma}{2} \pm\nu : \end{matrix} \right.$$

$\frac{1}{2},\ldots,\frac{1}{2}]$ $:-;\ldots;-;$

$\frac{1}{2},\ldots,\frac{1}{2}]$ $,[1+\beta_1+\ldots+\beta_r+\sigma:1,\ldots,1]$ $:-;\ldots;-;$

$$u_1(\alpha_1' x_1+\ldots+\alpha_r' x_r),\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r) \Bigg] \}$$

$$= \Phi_2^{(r)} [\beta_1,\ldots,\beta_r;\ \beta_1+\ldots+\beta_r;\ \frac{u_1}{\lambda},\ldots,\frac{u_r}{\lambda}],$$

where $\mathrm{Re}\ (\beta_1+\ldots+\beta_r+\sigma) > 2\ \mathrm{Re}\ |\nu| - 1$, and $K \neq 0$

(5.4.11) $S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}$ $\{F^{3:-;\ldots;-}_{2:2;\ldots;2}$ $\Bigg[ [\beta_1+\ldots+\beta_r:1,\ldots,1]$, $[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2} \pm \nu :$

$[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma) \pm \mu : \frac{1}{2},\ldots,\frac{1}{2}]$ $:-;\ldots;-;$

$\frac{1}{2},\ldots,\frac{1}{2}]$ $:[\beta_1:1],[\gamma_1:1];\ldots;[\beta_r:1],[\gamma_r:1];$

$$u_1(\alpha_1' x_1+\ldots+\alpha_r' x_r),\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r) \Bigg] \}$$

$$= \Psi_2^{(r)} [1+\beta_1+\ldots+\beta_r+\sigma;\ \gamma_1,\ldots,\gamma_r;\ \frac{u_1}{\lambda},\ldots,\frac{u_r}{\lambda}],$$

provided that $\mathrm{Re}\ (\beta_1+\ldots+\beta_r+\sigma) > 2\ \mathrm{Re}\ |\nu| - 1$ and $K \neq 0$

(5.4.12) $S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}$ $\{F^{3:-;\ldots;-}_{4:-;\ldots;-}$ $\Bigg[ [\beta_1+\ldots+\beta_r:1,\ldots,1]$, $[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2} \pm \mu :$

$$[1+\tfrac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu:\tfrac{1}{2},\ldots,\tfrac{1}{2}]:-;\ldots;-;$$

$$\tfrac{1}{2},\ldots,\tfrac{1}{2}],[\gamma:1,\ldots,1,0,\ldots,0],[\gamma':0,\ldots,0,1,\ldots,1]:-;\ldots;-;$$

$$\left.u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r),\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)\right]\}$$

$$= {}^{(k)}_{(1)}E_D^{(r)}\left(1+\beta_1+\ldots+\beta_r+\sigma,\beta_1,\ldots,\beta_r;\gamma,\gamma';\frac{u_1}{\lambda},\ldots,\frac{u_r}{\lambda}\right),$$

provided that Re $(\beta_1+\ldots+\beta_r+\sigma) > 2$ Re $|\nu| - 1$, $K \neq 0$

and if $|\frac{u_i}{\lambda_i}| < r_i$, $i = 1,\ldots,r$ then $r_1 = \ldots = r_k$, $r_{k+1}=\ldots=r_r$,

$r_k+r_r = 1$

(5.4.13) $$R^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{F^{3:1;\ldots;1}_{4:1;\ldots;1}\left[\begin{array}{l}[\beta_1+\ldots+\beta_r:1,\ldots,1],\\ [\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\pm\mu:\frac{1}{2},\ldots,\frac{1}{2}],\end{array}\right.$$

$$[1+\tfrac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu:\tfrac{1}{2},\ldots,\tfrac{1}{2}]:[\gamma_1:1];\ldots;[\gamma_r:1];$$

$$[\delta:1,\ldots,1,0,\ldots,0],[\delta':0,\ldots,0,1,\ldots,1]:[\beta_1:1];\ldots;[\beta_r:1];$$

$$\left.u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r),\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)\right]\}$$

$$= {}^{(k)}_{(1)}E_D^{(r)}\left(1+\beta_1+\ldots+\beta_r+\sigma,\gamma_1,\ldots,\gamma_r;\delta,\delta';\frac{u_1}{\lambda},\ldots,\frac{u_r}{\lambda}\right),$$

where all the conditions of (5.4.12) are satisfied

(5.4.14) $S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\ \{F^{4:-;\dots;-}_{3:-;\dots;-}\left[\begin{array}{l}[1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu:\\ [1+\beta_1+\dots+\beta_r+\sigma:\end{array}\right.$

$\frac{1}{2},\dots,\frac{1}{2}],\ [\gamma:1,\dots,1,0,\dots,0]\ ,\ [\gamma':0,\dots,0,1,\dots,1]:-;\dots;-;$

$1,\dots,1],\ [\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}\pm\nu:\frac{1}{2},\dots,\frac{1}{2}]:-;\dots;-;$

$$\left.u_1(\alpha_1'x_1+\dots+\alpha_r'x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)\right]\}$$

$$={}^{(k)}_{(2)}E^{(r)}_D\ (\gamma,\gamma',\beta_1,\dots,\beta_r;\beta_1+\dots+\beta_r;\ \frac{u_1}{\lambda},\dots,\frac{u_r}{\lambda}),$$

valid if $K\neq 0$, $\mathrm{Re}(\beta_1+\dots+\beta_r+\sigma) > 2\ \mathrm{Re}\ |\nu| - 1$ and if $|\frac{u_i}{\lambda}| < r_i$, $i=1,\dots,r$ then $r_1=\dots=r_k$, $r_{k+1}=\dots=r_r$, $r_k+r_r=r_k\cdot r_r$, the $r_1,\dots,r_r$ being the associated radii of convergence of the series ${}^{(k)}_{(2)}E^{(r)}_D$.

(5.4.15) $S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\ \{F^{5:-;\dots;-}_{2:2;\dots;2}\left[\begin{array}{l}[\beta_1+\dots+\beta_r:1,\dots,1]\ ,\\ [\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}\pm\nu:\frac{1}{2},\dots,\frac{1}{2}]:\end{array}\right.$

$[1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu:\frac{1}{2},\dots,\frac{1}{2}],[\delta:1,\dots,1,0,\dots,0],$

$[\beta_1:1]\ ,[\gamma_1:1]\ ;\dots;$

$[\delta':0,\dots,0,1,\dots,1]:-;\dots;-;$

$[\beta_r:1]\ ,[\gamma_r:1]\ ;$

$$u_1(\alpha_1' x_1+\cdots+\alpha_r' x_r),\ldots,u_r(\alpha_1^r x_1+\cdots+\alpha_r^r x_r)\Big]\Big\}$$

$$= {}^{(k)}_{(1)}E_C^{(r)}\ (\delta,\delta',1+\beta_1+\cdots+\beta_r+\sigma:\gamma_1,\ldots,\gamma_r\ ;\ \frac{u_1}{\lambda},\ldots,\frac{u_r}{\lambda}),$$

valid if $K \neq 0$, $\mathrm{Re}\,(\beta_1+\cdots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$ and if $|\frac{u_i}{\lambda_i}| < r_i$, $i = 1,\ldots,r$ then $(\sqrt{r_1}+\cdots+\sqrt{r_k})^2 + (\sqrt{r_{k+1}}+\cdots+\sqrt{r_r})^2 = 1$, where $r_1,\ldots,r_r$ being the associated radii of convergence of the series ${}^{(k)}_{(1)}E_C^{(r)}$.

(5.4.16) $$S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\ \Big\{{}_2F_4\Big[\begin{matrix}1+\frac{1}{2}(\beta_1+\cdots+\beta_r+\sigma)+\mu,\\ \frac{1+\beta_1+\cdots+\beta_r+\sigma}{2},\frac{2+\beta_1+\cdots+\beta_r+\sigma}{2}\end{matrix}$$

$$1+\frac{1}{2}(\beta_1+\cdots+\beta_r+\sigma)-\mu;$$

$$\frac{1+\beta_1+\cdots+\beta_r+\sigma}{2}+\nu,\ \frac{1+\beta_1+\cdots+\beta_r+\sigma}{2}-\nu;$$

$$\sum_{j=1}^{r} u_j(\alpha_1' x_1+\cdots+\alpha_r' x_r)\ \Big[\sum_{i=1}^{r}(\alpha_1^i x_1+\cdots+\alpha_r^i x_r)\Big]\Big\}$$

$$= \Phi_2^{(r)}(\beta_1,\ldots,\beta_r;\beta_1+\cdots+\beta_r;\ \frac{4u_1}{\lambda^2},\ldots,\frac{4u_r}{\lambda^2}),$$

provided that $\mathrm{Re}\,(\beta_1+\cdots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$ and $K \neq 0$

(5.4.17) $$S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\ \Big\{F^{3:-;\ldots;-}_{3:1;\ldots;1}\Big[\begin{matrix}[\beta_1+\cdots+\beta_r:1,\ldots,1],\\ [\frac{2+\beta_1+\cdots+\beta_r+\sigma}{2}:1,\ldots,1],\end{matrix}$$

$[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)+\mu:1,\ldots,1]$ $,[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)-\mu:1,\ldots,1]$ :

$[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}+\nu:1,\ldots,1]$, $[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}-\nu:1,\ldots,1]$,

$-;\ldots;-;$

$[\gamma_1:1]$ ; $[\gamma_r:1]$ ;

$$\left.\frac{u_1(\alpha_1^1x_1+\ldots+\alpha_r^1x_r)}{\sum_{j=1}^{r}(\alpha_1^jx_1+\ldots+\alpha_r^jx_r)},\ldots,\frac{u_r(\alpha_1^rx_1+\ldots+\alpha_r^rx_r)}{\sum_{j=1}^{r}(\alpha_1^jx_1+\ldots+\alpha_r^jx_r)}\right]\}$$

$$= F_A^{(r)}\left(\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2},\beta_1,\ldots,\beta_r;\gamma_1,\ldots,\gamma_r;\ \frac{4u_1}{\lambda^2},\ldots,\frac{4u_r}{\lambda^2}\right),$$

where $\operatorname{Re}(\beta_1+\ldots+\beta_r+\sigma) > 2\operatorname{Re}|\nu| - 1$ and $|\frac{u_1}{\lambda^2}|+\ldots+|\frac{u_r}{\lambda^2}| < \frac{1}{4}$ and $K \neq 0$

(5.4.18) $S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{F^{3:-;\ldots;-}_{2:1,\ldots;1}\left[\begin{array}{l}[\beta_1+\ldots+\beta_r:1,\ldots,1],\\ [1+\beta_1+\ldots+\beta_r+\sigma:2,\ldots,2],\end{array}\right.$

$[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu:1,\ldots,1]:-;\ldots;-;$

$[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}-\nu:1,\ldots,1]:[\gamma_1:1];\ldots;[\gamma_r:1]$;

$$\left.\frac{u_1(\alpha_1^1x_1+\ldots+\alpha_r^1x_r)}{\sum_{j=1}^{r}(\alpha_1^jx_1+\ldots+\alpha_r^jx_r)},\ldots,\frac{u_r(\alpha_1^rx_1+\ldots+\alpha_r^rx_r)}{\sum_{j=1}^{r}(\alpha_1^jx_1+\ldots+\alpha_r^jx_r)}\right]\}$$

$$= F_A^{(r)}\left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}+\nu, \beta_1,\dots,\beta_r; \gamma_1,\dots,\gamma_r; \frac{u_1}{\lambda^2},\dots,\frac{u_r}{\lambda^2}\right),$$

provided that $\mathrm{Re}\,(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$,

$\left|\frac{u_1}{\lambda^2}\right| + \dots + \left|\frac{u_r}{\lambda^2}\right| < 1$ and $K \neq 0$

$$(5.4.19)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\Big\{F^{2:1;\dots;1}_{3:-;\dots;-}\left[\begin{array}{l}[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu: \\ {[1+\beta_1+\dots+\beta_r+\sigma:2,\dots,2]},\end{array}\right.$$

$$1,\dots,1]: [\gamma_1:1];\dots;[\gamma_r:1];$$

$$\left[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}\pm\nu:1,\dots,1\right]:-;\dots;-;$$

$$\left.\begin{array}{l} \dfrac{u_1(\alpha_1^1 x_1+\dots+\alpha_r^1 x_r)}{\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r)},\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r) \\ \qquad\qquad \sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\end{array}\right]\Big\}$$

$$= F_B^{(r)}\left(\beta_1,\dots,\beta_r,\gamma_1,\dots,\gamma_r; \beta_1+\dots+\beta_r; \frac{u_1}{\lambda^2},\dots,\frac{u_r}{\lambda^2}\right),$$

provided that $\mathrm{Re}\,(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$, $K \neq 0$ and all

$\left|\frac{u_i}{\lambda^2}\right| < 1$, $i = 1,\dots,r$.

$$(5.4.20)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\Big\{F^{2:1;\dots;1}_{3:-;\dots;-}\left[\begin{array}{l}[\beta_1+\dots+\beta_r:1,\dots,1], \\ {[1+\beta_1+\dots+\beta_r+\sigma:2,\dots,2]},\end{array}\right.$$

$[1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma)+\mu:1,\dots,1]:[\gamma_1:1];\dots;[\gamma_r:1];$

$[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}\pm\nu:1,\dots,1]:-;\dots;-;$

$$u_1(\alpha_1'x_1+\dots+\alpha_r'x_r)\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\right]\}$$

$$=F_B^{(r)}(\beta_1,\dots,\beta_r,\gamma_1,\dots,\gamma_r;1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)-\mu;\frac{u_1}{\lambda^2},\dots,\frac{u_r}{\lambda^2}),$$

provided that all the conditions of (5.4.19) are satisfied.

(5.4.21) $$S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\{F^{3:-;\dots;-}_{1:2;\dots;2}\left[\begin{array}{l}[\beta_1+\dots+\beta_r:1,\dots,1],\\ [1+\beta_1+\dots+\beta_r+\sigma:2,\dots,2]:\end{array}\right.$$

$[1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu:1,\dots,1]:-;\dots;-;$

$[\beta_1:1],[\gamma_1:1];\dots;[\beta_r:1],[\gamma_r:1];$

$$u_1(\alpha_1'x_1+\dots+\alpha_r'x_r)\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\right]\}$$

$$=F_C^{(r)}(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}+\nu,\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}-\nu;\gamma_1,\dots,\gamma_r;\frac{u_1}{\lambda^2},\dots,\frac{u_r}{\lambda^2}),$$

where Re $(\beta_1+\dots+\beta_r+\sigma) > 2$ Re $|\nu| - 1$, $\sum_{i=1}^{r} |(\frac{u_i}{\lambda^2})^{1/2}| < 1$ and $K \neq 0$

$$(5.4.22)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\ \{F^{3:-;\dots;-}_{2:2;\dots;2}\left[\begin{array}{l}[\beta_1+\dots+\beta_r:1],\\ [\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}\pm\nu:1,\dots,1]:\end{array}\right.$$

$$[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu:1,\dots,1]\ :-;\dots;-;$$

$$[\beta_1:1],[\gamma_1:1];\dots;[\beta_r:1],[\gamma_r:1];$$

$$u_1(\alpha_1' x_1+\dots+\alpha_r' x_r)\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\right]\}$$

$$= F_C^{(r)}\left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2},\ \frac{2+\beta_1+\dots+\beta_r+\sigma}{2};\ \gamma_1,\dots,\gamma_r;\ \frac{4u_1}{\lambda^2},\dots,\ \frac{4u_r}{\lambda^2}\right)$$

valid if Re $(\beta_1+\dots+\beta_r+\sigma) > 2$ Re $|\nu| - 1$, $\sum_{i=1}^{r} |(\frac{u_i}{\lambda^2})^2| < \frac{1}{2}$ and $K \neq 0$.

$$(5.4.23)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\ \{F^{3:-;\dots;-}_{2:2;\dots;2}\left[\begin{array}{l}[\beta_1+\dots+\beta_r:1,\dots,1],\\ [\frac{2+\beta_1+\dots+\beta_r+\sigma}{2}:1,\dots,1],\end{array}\right.$$

$$[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu:1,\dots,1]\ :-;\dots;-;$$

$$[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}+\nu:1,\dots,1]:[\beta_1:1],[\gamma_1:1];\dots;[\beta_r:1],[\gamma_r:1];$$

$$u_1(\alpha_1^1 x_1+\dots+\alpha_r^1 x_r) \sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r)\right] \}$$

$$= F_C^{(r)}\left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}, \frac{1+\beta_1+\dots+\beta_r+\sigma}{2}-\nu;\ \gamma_1,\dots,\gamma_r;\ \frac{4u_1}{\lambda^2},\dots,\frac{4u_r}{\lambda^2}\right),$$

provided that all the conditions of (5.4.22) are satisfied

$$(5.4.24)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\ \{{}_2F_3\left[\begin{matrix}1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma)+\mu, \\ \frac{1+\beta_1+\dots+\beta_r+\sigma}{2}, \frac{2+\beta_1+\dots+\beta_r+\sigma}{2},\end{matrix}\right.$$

$$1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma)-\mu;$$

$$\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}+\nu;$$

$$\left.\sum_{j=1}^{r} u_j(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\ \left[\sum_{i=1}^{r} (\alpha_1^i x_1+\dots+\alpha_r^i x_r)\right]\right] \}$$

$$= F_D^{(r)}\left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}-\nu, \beta_1,\dots,\beta_r; \beta_1+\dots+\beta_r;\ \frac{4u_1}{\lambda^2},\dots,\frac{4u_r}{\lambda^2}\right),$$

valid if $\mathrm{Re}\,(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$, $K \neq 0$ and $\left|\frac{u_i}{\lambda^2}\right| < \frac{1}{4}$, $i = 1,\dots,r$

$$(5.4.25)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\ \{{}_2F_3\left[\begin{matrix}\beta_1+\dots+\beta_r, 1+\mu+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma); \\ \frac{2+\beta_1+\dots+\beta_r+\sigma}{2}, \frac{1+\beta_1+\dots+\beta_r+\sigma}{2}+\nu,\end{matrix}\right.$$

$$\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}-\nu;$$

$$\left.\left.\sum_{j=1}^{r} u_j(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\left[\sum_{i=1}^{r}(\alpha_1^i x_1+\dots+\alpha_r^i x_r)\right]\right]\right\}$$

$$= F_D^{(r)}\left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2},\beta_1,\dots,\beta_r;1-\mu+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma);\ \frac{4u_1}{\lambda^2},\dots,\frac{4u_r}{\lambda^2}\right),$$

where all the conditions of (5.4.24) are satisfied.

(5.4.26) $$S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\left\{{}_2F_4\left[\begin{matrix}\beta_1+\dots+\beta_r,1+\frac{1}{2}(\beta_1+\dots+\beta_r+\sigma)-\mu;\\ \frac{1+\beta_1+\dots+\beta_r+\sigma}{2},\ \frac{2+\beta_1+\dots+\beta_r+\sigma}{2},\end{matrix}\right.\right.$$

$$\left.\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}+\nu,\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}-\nu;\right.$$

$$\left.\left.\sum_{j=1}^{r} u_j(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\left[\sum_{i=1}^{r}(\alpha_1^i x_1+\dots+\alpha_r^i x_r)\right]\right]\right\}$$

$$= \Phi_2^{(r)}\left[\beta_1,\dots,\beta_r;1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)+\mu;\ \frac{4u_1}{\lambda^2},\dots,\frac{4u_r}{\lambda^2}\right],$$

provided that $\mathrm{Re}\,(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$ and $K \neq 0$

(5.4.27) $$S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\left\{F^{3:-;\dots;-}_{2:2;\dots;2}\left[\begin{matrix}[\beta_1+\dots+\beta_r:1,\dots,1],\\ [1+\beta_1+\dots+\beta_r+\sigma:2,\dots,2],\end{matrix}\right.\right.$$

$$[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu:1,\dots,1]:-;\dots;-;$$

$$\left[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}+\nu:1,\dots,1\right]:[\beta_1:1],[\gamma_1:1];\dots;[\beta_r:1],[\gamma_r:1];$$

$$u_1(\alpha_1' x_1+\dots+\alpha_r' x_r) \sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r) \Bigg/ \sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r) \Bigg] \}$$

$$= \Psi_2^{(r)} \left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2} - \nu;\ \gamma_1,\dots,\gamma_r;\ \frac{u_1}{\lambda^2},\dots,\frac{u_r}{\lambda^2}\right),$$

where $\mathrm{Re}\,(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$ and $K \neq 0$

$$(5.4.28)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma} \{F^{3:-;\dots;-}_{3:2;\dots;2} \left[\begin{array}{l} [\beta_1+\dots+\beta_r:1,\dots,1], \\ [\frac{2+\beta_1+\dots+\beta_r+\sigma}{2}:1,\dots,1], \end{array}\right.$$

$$[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma) \pm\mu:1,\dots,1] :-;\dots;-;$$

$$[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2} \pm\nu:1,\dots,1] :[\beta_1:1],[\gamma_1:1];\dots;[\beta_r:1],[\gamma_r:1];$$

$$u_1(\alpha_1' x_1+\dots+\alpha_r' x_r) \sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r) \Bigg/ \sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r) \Bigg] \}$$

$$= \Psi_2^{(r)} \left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2};\ \gamma_1,\dots,\gamma_r;\ \frac{4u_1}{\lambda^2},\dots,\frac{4u_r}{\lambda^2}\right),$$

provided that $\mathrm{Re}\,(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$ and $K \neq 0$

$$(5.4.29)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma} \{F^{3:-;\dots;-}_{3:2;\dots;2} \left[\begin{array}{l} [\beta_1+\dots+\beta_r:1,\dots,1], \\ [\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}:1,\dots,1], \end{array}\right.$$

$$[1 + \tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma) \pm \mu :1,\dots,1] :-;\dots;-;$$

$$[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2} \pm \nu :1,\dots,1]:[\beta_1:1],[\gamma_1:1];\dots;[\beta_r:1],[\gamma_r:1];$$

$$u_1(\alpha_1' x_1+\dots+\alpha_r' x_r) \sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r) \right] \}$$

$$= \Psi_2^{(r)} \left(\frac{2+\beta_1+\dots+\beta_r+\sigma}{2} ;\gamma_1,\dots,\gamma_r ; \frac{4u_1}{\lambda^2},\dots,\frac{4u_r}{\lambda^2}\right),$$

where $\mathrm{Re}\,(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$ and $K \neq 0$

$$(5.4.30) \quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r,\sigma} \{F^{3:-;\dots;-}_{4:-;\dots;-} \left[\begin{matrix}[\beta_1+\dots+\beta_r:1,\dots,1], \\ [\gamma:1,\dots,1,0,\dots,0],\end{matrix}\right.$$

$$[1 + \tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma) \pm \mu :1,\dots,1] :-;\dots;-;$$

$$[\gamma',0,\dots,0,1,\dots,1],[1+\beta_1+\dots+\beta_r+\sigma:2,\dots,2],[\frac{2+\beta_1+\dots+\beta_r+\sigma}{2}$$

$$- \nu:1,\dots,1]:-;\dots;-;$$

$$u_1(\alpha_1' x_1+\dots+\alpha_r' x_r) \sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r) \right] \}$$

$$= {}^{(k)}_{(1)}E_D^{(r)} \left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2} + \nu,\beta_1,\dots,\beta_r;\gamma,\gamma' ; \frac{u_1}{\lambda^2},\dots,\frac{u_r}{\lambda^2}\right),$$

provided that Re $(\beta_1+\dots+\beta_r+\sigma) > 2$ Re $|\nu| - 1$, $K \neq 0$, and $|\frac{u_i}{\lambda^2}| < r_i$ with $r_1 = \dots = r_k$, $r_{k+1} = \dots = r_r$, $r_k + r_r = 1$, $r_i$, $i = 1,\dots,r$, being the associated radii of convergence of the series ${}^{(k)}_{(1)}E_D^{(r)}$.

$$(5.4.31)\quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\ \{F^{3:-;\dots;-}_{5:-;\dots;-}\left[\begin{array}{l}[\beta_1+\dots+\beta_r:1,\dots,1],\\ [\gamma:1,\dots,1,0,\dots,0],\end{array}\right.$$

$$[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)\pm\mu:1,\dots,1]:-;\dots;-;$$

$$[\gamma':0,\dots,0,1,\dots,1],[\frac{2+\beta_1+\dots+\beta_r+\sigma}{2}:1,\dots,1],[\frac{1+\beta_1+\dots+\beta_r+\sigma}{2}\pm\nu:$$

$$1,\dots,1]:-;\dots;-;$$

$$u_1(\alpha_1'x_1+\dots+\alpha_r'x_r)\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r}(\alpha_1^j x_1+\dots+\alpha_r^j x_r)\right]\}$$

$$= {}^{(k)}_{(1)}E_D^{(r)}\left(\frac{1+\beta_1+\dots+\beta_r+\sigma}{2},\beta_1,\dots,\beta_r;\gamma,\gamma';\ \frac{4u_1}{\lambda^2},\dots,\frac{4u_r}{\lambda^2}\right),$$

provided that Re $(\beta_1+\dots+\beta_r+\sigma) > 2$ Re $|\nu| - 1$, $K \neq 0$ and $|\frac{u_i}{\lambda^2}| < \frac{r_i}{4}$ with $r_1 = \dots = r_k$, $r_{k+1} = \dots = r_r$, $r_k + r_r = 1$, where $r_i$, $i = 1,\dots,r$ being the associated radii of convergence of the series ${}^{(k)}_{(1)}E_D^{(r)}$.

$$(5.4.32)\quad S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\ \{F^{3:-;\ldots;-}_{5:-;\ldots;-}\left[\begin{array}{l}[\beta_1+\ldots+\beta_r:1,\ldots,1],\\ [\gamma:1,\ldots,1,0,\ldots,0],\end{array}\right.$$

$$[1+\tfrac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu:1,\ldots,1]:-;\ldots;-;$$

$$[\gamma':0,\ldots,0,1,\ldots,1],[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}:1,\ldots,1],\ \frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\pm\nu:$$

$$1,\ldots,1]:-;\ldots;-;$$

$$u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)\sum_{j=1}^{r}(\alpha_1^j x_1+\ldots+\alpha_r^j x_r),\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r}(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)\right]\}$$

$$={}^{(k)}_{(1)}E^{(r)}_{D}\left(\frac{2+\beta_1+\ldots+\beta_r+\sigma}{2},\beta_1,\ldots,\beta_r;\gamma,\gamma';\ \frac{4u_1}{\lambda^2},\ldots,\frac{4u_r}{\lambda^2}\right),$$

provided that all the conditions of (5.4.31) are satisfied.

$$(5.4.33)\quad S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\ \{F^{4:-;\ldots;-}_{3:-;\ldots;-}\left[\begin{array}{l}[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu:1,\ldots,1],\\ [1+\beta_1+\ldots+\beta_r+\sigma:2,\ldots,2],\end{array}\right.$$

$$[\gamma:1,\ldots,1,0,\ldots,0],\ [\gamma':0,\ldots,0,1,\ldots,1]:-;\ldots;-;$$

$$[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\pm\nu:1,\ldots,1]:-;\ldots;-;$$

$$u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)\sum_{j=1}^{r}(\alpha_1^j x_1+\ldots+\alpha_r^j x_r),\ldots,u_r(\alpha_r^r x_r+\ldots+\alpha_r^r x_r)$$

$$\sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r)$$

$$= {}^{(k)}_{(2)}E_D^{(r)} \left(\gamma,\gamma',\beta_1,\dots,\beta_r;\beta_1+\dots+\beta_r\,;\, \frac{u_1}{\lambda^2},\dots,\frac{u_r}{\lambda^2}\right),$$

valid if $\mathrm{Re}\,(\beta_1+\dots+\beta_r+\sigma) > 2\,\mathrm{Re}\,|\nu| - 1$, $K \neq 0$ and $|\frac{u_i}{\lambda^2}| < r_i$, with $r_1 =\dots= r_k$, $r_{k+1} =\dots= r_r$, $r_k + r_r = r_k \cdot r_r$ where $r_i$, $i = 1,\dots,r$, are the associated radii of convergence of ${}^{(k)}_{(2)}E_D^{(r)}$.

$$(5.4.34) \quad S^{\lambda,\mu,\nu}_{\beta_1,\dots,\beta_r;\sigma}\left\{F^{4:-;\dots;-}_{3:-;\dots;-}\left[\begin{matrix}[1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)+\mu: \\ [1+\beta_1+\dots+\beta_r+\sigma:2,\dots,2],\end{matrix}\right.\right.$$

$$1,\dots,1],[\beta_1+\dots+\beta_r:1,\dots,1],[\gamma:1,\dots,1,0,\dots,0]$$

$$\left[\frac{1+\beta_1+\dots\beta_r+\sigma}{2} \pm \nu:1,\dots,1\right]:$$

$$[\gamma':0,\dots,0,1,\dots,1]:-;\dots;-;$$

$$-;\dots;-;$$

$$\left.\left. u_1(\alpha_1' x_1+\dots+\alpha_r' x_r)\,\frac{\displaystyle\sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r)}{\displaystyle\sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r)},\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)\right]\right\}$$

$$= {}^{(k)}_{(2)}E_D^{(r)} \left(\gamma,\gamma',\beta_1,\dots,\beta_r;1+\tfrac{1}{2}(\beta_1+\dots+\beta_r+\sigma)-\mu\,;\, \frac{u_1}{\lambda^2},\dots,\frac{u_r}{\lambda^2}\right),$$

provided that all the conditions of (5.4.33) are satisfied

(5.4.35) $S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{F^{5:-;\ldots;-}_{2:2;\ldots;2}\left[\begin{array}{l}[\beta_1+\ldots+\beta_r:1,\ldots,1]\,, \\ [1+\beta_1+\ldots+\beta_r+\sigma:2,\ldots,2]\,,\end{array}\right.$

$[1+\frac{(\beta_1+\ldots+\beta_r+\sigma)}{2}\pm\mu:1,\ldots,1],\ [\delta:1,\ldots,1,0,\ldots,0]\,,$

$[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}+\nu:1,\ldots,1]\,,\ [\beta_1:1]\,,\ [\gamma_1:1]\,;\ldots;$

$[\delta':0,\ldots,0,1,\ldots,1]:-;\ldots;-;$

$[\beta_r:1]\,,\ [\gamma_r:1]\,;$

$u_1(\alpha'_1x_1+\ldots+\alpha'_rx_r)\sum_{j=1}^{r}(\alpha^j_1x_1+\ldots+\alpha^j_rx_r),\ldots,u_r(\alpha^r_1x_1+\ldots+\alpha^r_rx_r)$

$\left.\sum_{j=1}^{r}(\alpha^j_1x_1+\ldots+\alpha^j_rx_r)\right]\}$

$={}^{(k)}_{(1)}E^{(r)}_{C}\ (\delta,\delta',\ \frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}-\nu\,;\gamma_1,\ldots,\gamma_r;\ \frac{u_1}{\lambda^2},\ldots,\frac{u_r}{\lambda^2}),$

where $\mathrm{Re}\,(\beta_1+\ldots+\beta_r+\sigma)>2\,\mathrm{Re}\,|\nu|-1$, $K\neq 0$ and $|\frac{u_i}{\lambda^2}|<r_i$ with $(\sqrt{r_1}+\ldots+\sqrt{r_k})^2+(\sqrt{r_{k+1}}+\ldots+\sqrt{r_r})^2=1$, $r_i$, $i=1,\ldots,r$, being the associated radii of convergence of the series ${}^{(k)}_{(1)}E^{(r)}_{C}$.

(5.4.36) $S^{\lambda,\mu,\nu}_{\beta_1,\ldots,\beta_r;\sigma}\{F^{5:-;\ldots;-}_{3:2;\ldots;2}\left[\begin{array}{l}[\beta_1+\ldots+\beta_r:1,\ldots,1]\,, \\ [\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}:1,\ldots,1]\,,\end{array}\right.$

$[1+\frac{1}{2}(\beta_1+\ldots+\beta_r+\sigma)\pm\mu:1,\ldots,1],[\delta:1,\ldots,1,0,\ldots,0]\,,$

$[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2}\pm\nu:1,\ldots,1]:[\beta_1:1]\,,\ [\gamma_1:1]\,;\ldots;$

$[\delta' : 0, \ldots, 0, 1, \ldots, 1] : -; \ldots ; -;$

$[\beta_r : 1]\ , [\gamma_r : 1]\ ;$

$$u_1(\alpha_1' x_1 + \ldots + \alpha_r' x_r) \sum_{j=1}^{r} (\alpha_1^j x_1 + \ldots + \alpha_r^j x_r), \ldots, u_r(\alpha_1^r x_1 + \ldots + \alpha_r^r x_r)$$

$$\left. \sum_{j=1}^{r} (\alpha_1^j x_1 + \ldots + \alpha_r^j x_r) \right] \}$$

$$= {}^{(k)}_{(1)}E_C^{(r)} \left(\delta, \delta', \frac{1+\beta_1+\ldots+\beta_r+\sigma}{2} ; \gamma_1, \ldots, \gamma_r ; \frac{4u_1}{\lambda^2}, \ldots, \frac{4u_r}{\lambda^2}\right),$$

provided that Re $(\beta_1 + \ldots + \beta_r + \sigma) > 2\,|\nu| - 1$, $K \neq 0$, and $\left|\frac{u_i}{\lambda^2}\right| < \frac{r_i}{4}$, with $(\sqrt{r_1} + \ldots + \sqrt{r_k})^2\ (\sqrt{r_{k+1}} + \ldots + \sqrt{r_r})^2 = 1$, $r_i$, $i = 1, \ldots, r$, being associated radii of convergence of the series ${}^{(k)}_{(1)}E_C^{(r)}$.

$$(5.4.37) \quad S^{\lambda, \mu, \nu}_{\beta_1, \ldots, \beta_r; \sigma} \{ F^{5:-;\ldots;-}_{3:2;\ldots;-} \left[ \begin{array}{l} [\beta_1 + \ldots + \beta_r : 1, \ldots, 1]\ , \\ \\ [\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2} : 1, \ldots, 1]\ , \end{array} \right.$$

$[1 + \frac{1}{2}(\beta_1 + \ldots + \beta_r + \sigma) \pm \mu : 1, \ldots, 1]\ , [\delta : 1, \ldots, 1, 0, \ldots, 0]\ ,$

$[\frac{1+\beta_1+\ldots+\beta_r+\sigma}{2} \pm \nu : 1, \ldots, 1] : [\beta_1 : 1]\ , [\gamma_1 : 1]\ ; \ldots ;$

$[\delta' : 0, \ldots, 0, 1, \ldots, 1] : -; \ldots ; -;$

$[\beta_r : 1]\ , [\gamma_r : 1]\ ;$

$$u_1(\alpha_1' x_1+\dots+\alpha_r' x_r) \sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r),\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)$$

$$\left.\sum_{j=1}^{r} (\alpha_1^j x_1+\dots+\alpha_r^j x_r)\right]\}$$

$$= {}^{(k)}_{(1)}E_C^{(r)}\left(\delta,\delta', \frac{2+\beta_1+\dots+\beta_r+\sigma}{2}; \gamma_1,\dots,\gamma_r; \frac{4u_1}{\lambda^2},\dots,\frac{4u_r}{\lambda^2}\right),$$

valid if all the conditions of $B_{31}$ are satisfied.

Similarly for particular interest specialising the variables and applying the same techniques we can obtain generalized multiple Whittaker transforms of hypergeometric functions of different variables.

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CHAPTER VI

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# ANOTHER MULTIDIMENSIONAL WHITTAKER TRANSFORM OF MULTIPLE HYPERGEOMETRIC FUNCTIONS IN SEVERAL VARIABLES

6.1. Introduction. Chandel [1] introduced n-dimensional Laplacian integral operator to evaluate multiple integrals involving Lauricella functions [9] of several variables. Further Chandel [2] studied Eulerian integral transforms of hypergeometric functions of several variables of Lauricella [9] and of their confluent forms with the help of fractional integration. Again in an extension Chandel [3] studied Laplacian and Eulerian integral transforms of multiple hypergeometric functions of several variables defined by Exton [7] and Chandel [3]. Recently, Chandel and Dwivedi [4,5,6] introduced and studied three types of multiple Whittaker transforms to evaluate certain multiple integrals involving hypergeometric functions of Srivastava and Daoust [10] Lauricella [9], Exton [7], Chandel [3] and also involving their confluent forms. In previous chapter V, we have studied new multidimensional Whittaker transforms of hypergeometric functions in several variables, now in this chapter we further introduce the following new multiple Whittaker transform :

$$(6.1.1)\quad M^{\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r}_{\beta_1,\ldots,\beta_r;\lambda_1,\ldots,\lambda_r}\{\ \} \equiv M\{\ \}$$

$$= K \prod_{j=1}^{r} \frac{2\,\lambda_j^{\beta_j}\,\Gamma(1+\frac{1}{2}\beta_j \pm \mu_j)}{\Gamma(1+\beta_j)\,\Gamma(\frac{1+\beta_j}{2} \pm \nu_j)} \int_0^\infty\int_0^\infty \prod_{j=1}^{r} (\alpha_1^j x_1+\ldots+\alpha_r^j x_r)^{\beta_j-1}$$

$$W_{\mu_j,\nu_j}[\lambda_j(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)]\; W_{\mu_j,-\nu_j}[\lambda_j(\alpha_1^j x_1+\ldots+\alpha_r^j x_r)]\;\{\ \}$$

$$dx_1 \ldots dx_r,$$

provided that $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$, $j = 1,\ldots,r$ and

$$K = \begin{vmatrix} \alpha_1' & \alpha_2' & \ldots & \alpha_r' \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_r^2 \\ \ldots & \ldots\ldots & & \ldots \\ \alpha_1^r & \alpha_2^r & \ldots & \alpha_r^r \end{vmatrix} \neq 0,$$

to evaluate certain multiple integrals involving Srivastava and Daoust function [10]. For particular interest some special cases will also be discussed to obtain generalized multiple Whittaker transforms involving multiple hypergeometric functions of several variables defined by Chandel [3], Exton [7,8] and Lauricella [9].

6.2 Whittaker transform of Srivastava and Daoust function. In this section we shall establish some properties of the Whittaker transform and their applications will be made in obtaining multiple Whittaker transform of Srivastava and Daoust function of several variables. From (6.1.1) it is

clear that

$$(6.2.1)\quad M^{\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r}_{\beta_1,\ldots,\beta_r;\lambda_1,\ldots,\lambda_r} \equiv M^{-\mu_1,\ldots,-\mu_r;\nu_1,\ldots,\nu_r}_{\beta_1,\ldots,\beta_r;\lambda_1,\ldots,\lambda_r}$$

$$\equiv M^{\mu_1,\ldots,\mu_r;-\nu_1,\ldots,-\nu_r}_{\beta_1,\ldots,\beta_r;\lambda_1,\ldots,\lambda_r} \equiv M^{-\mu_1,\ldots,-\mu_r;-\nu_1,\ldots,-\nu_r}_{\beta_1,\ldots,\beta_r;\lambda_1,\ldots,\lambda_r}$$

$$\equiv M^{-\mu_1,\ldots,-\mu_r;-\nu_1,\ldots,-\nu_r}_{-\beta_1,\ldots,-\beta_r;\lambda_1,\ldots,\lambda_r} \equiv M^{-\mu_1,\ldots,-\mu_r;-\nu_1,\ldots,-\nu_r}_{-\beta_1,\ldots,-\beta_r;-\lambda_1,\ldots,-\lambda_r}$$

$$(6.2.2)\quad M^{\mu_1,\ldots,\mu_r;\nu_1,\ldots,\nu_r}_{\beta_1,\ldots,\beta_r;\lambda_1,\ldots,\lambda_r}\{1\} = 1,$$

$$(6.2.3)\quad M\{(\alpha_1 x_1+\ldots+\alpha_r' x_r)^{2m_1} \ldots (\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^{2m_r}\}$$

$$= \prod_{j=1}^{r} \frac{(\beta_1+1)_{2m_j}\left(\frac{1+\beta_j}{2} \pm \nu_j\right)_{m_j}}{\lambda_j^{2m_j}\left(1+\frac{1}{2}\beta_j \pm \mu_j\right)_{m_j}}$$

$$= \prod_{j=1}^{r} \frac{2^{2m_j}\left(\frac{\beta_j+1}{2}\right)_{m_j}\left(\frac{\beta_j}{2}+1\right)_{m_j}\left(\frac{1+\beta_j}{2} \pm \nu_j\right)_{m_j}}{\lambda_j^{2m_j}\left(1+\frac{1}{2}\beta_j \pm \mu_j\right)_{m_j}}$$

$$(6.2.4)\quad M\{(\alpha_1' x_1+\ldots+\alpha_r' x_r)^{2m_1\xi_1} \ldots (\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^{2m_r\xi_r}\}$$

$$= \prod_{j=1}^{r} \frac{(1+\beta_j)_{2m_j\xi_j}\left(\frac{1+\beta_j}{2} \pm \nu_j\right)_{m_j\xi_j}}{\lambda_j^{2m_j\xi_j}\left(1+\frac{1}{2}\beta_j \pm \mu_j\right)_{m_j\xi_j}}.$$

An appeal to (6.2.4) gives the following Whittaker transform for generalized multiple hypergeometric function of Srivastava and Daoust [10].

(6.2.5) $$M\{F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\; u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)^{2\xi_1},\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^{2\xi_r}\right]\}$$

$$= F^{A:B'+3;\ldots;B^{(r)}+3}_{C:D'+2;\ldots;D^{(r)}+2}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\phi'],[\beta_1+1:2\xi_1],\left[\frac{1+\beta_1}{2}\pm\nu_1:\xi_1\right];\ldots;[(b^{(r)}):\phi^{(r)}],[\beta_r+1:2\xi_r],\left[\frac{1+\beta_r}{2}\pm\nu_r:\xi_r\right];\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'],[1+\tfrac{1}{2}\beta_1\pm\mu_1:\xi_1];\ldots;[(d^{(r)}):\delta^{(r)}],[1+\tfrac{1}{2}\beta_r\pm\mu_r:\xi_r];\end{matrix}\;\frac{u_1}{\lambda_1^{2\xi_1}},\ldots,\frac{u_r}{\lambda_r^{2\xi_r}}\right]$$

provided that $\operatorname{Re}(\beta_j) > 2\operatorname{Re}|\nu_j| - 1$, $j = 1,\ldots,r$,

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)}-2\xi_i > 0 \text{ and}$$

$$K = \begin{vmatrix} \alpha_1' & \alpha_2' & \cdots & \alpha_r' \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_r^2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^r & \alpha_2^r & \cdots & \alpha_r^r \end{vmatrix} \neq 0.$$

6.3 Special Cases. For particular interest here we establish the following special cases of (6.2.5)

$$(6.3.1)\quad M\{F^{1:1;\ldots;1}_{-:3;\ldots;3}\left[\begin{array}{l}[a:1,\ldots,1]:[1+\tfrac{1}{2}\beta_1+\mu_1:1]\ ;\\ -:[1+\tfrac{\beta_1}{2}:1],[\tfrac{1+\beta_1}{2}\pm\nu_1:1]\ ;\end{array}\right.$$

$$\ldots;[1+\tfrac{1}{2}\beta_r+\mu_r:1]\ ;$$

$$\ldots;[1+\tfrac{\beta_r}{2}:1],[\tfrac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left. u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= F_A^{(r)}\left[a,\frac{1+\beta_1}{2},\ldots,\frac{1+\beta_r}{2};1+\tfrac{1}{2}\beta_1-\mu_1,\ldots,1+\tfrac{1}{2}\beta_r-\mu_r;\frac{4u_1}{\lambda_1^2},\ldots,\frac{4u_r}{\lambda_r^2}\right]$$

provided that $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1;\ j = 1,\ldots,r$, $K \neq 0$ and

$$\sum_{i=1}^{r} \left|\frac{u_i}{\lambda_i^2}\right| < 1.$$

$$(6.3.2)\quad M\{F^{1:1;\ldots;1}_{-:3;\ldots;3}\left[\begin{array}{l}[a:1,\ldots,1]:[1+\tfrac{1}{2}\beta_1+\mu_1:1];\ \ldots\\ -:[\tfrac{1+\beta_1}{2}:1],[\tfrac{1+\beta_1}{2}\pm\nu_2:1]\ ;\ \ldots\end{array}\right.$$

$\ldots; [1 + \frac{1}{2}\beta_r + \mu_r:1];$

$\ldots; [\frac{1+\beta_r}{2}], [\frac{1+\beta_r}{2} \pm \nu_r:1];$

$$\left. u_1(\alpha_1' x_1 + \ldots + \alpha_r' x_r)^2, \ldots, u_r(\alpha_1^r x_1 + \ldots + \alpha_r^r x_r)^2 \right] \}$$

$$= F_A^{(r)} \left[ a, 1 + \frac{\beta_1}{2}, \ldots, 1 + \frac{\beta_r}{2}; 1 + \frac{1}{2}\beta_1 - \mu_1, \ldots, 1 + \frac{1}{2}\beta_r - \mu_r; \frac{4u_1}{\lambda_1^2}, \ldots, \frac{4u_r}{\lambda_r^2} \right]$$

provided that all conditions of (6.3.1) are satisfied.

$$(6.3.3) \quad M\{F^{1:1;\ldots;1}_{-:2;\ldots;2} \left[ \begin{matrix} [a:1,\ldots,1]:[1+\frac{1}{2}\beta_1+\mu_1:1];\ldots \\ -:[1+\beta_1:2], [\frac{1+\beta_1}{2}+\nu_1:1];\ldots \end{matrix} \right.$$

$\ldots; [1 + \frac{1}{2}\beta_r + \mu_r:1];$

$\ldots; [1 + \beta_r:2], [\frac{1+\beta_r}{2} + \nu_r:1];$

$$\left. u_1(\alpha_1' x_1 + \ldots + \alpha_r' x_r)^2, \ldots, u_r(\alpha_1^r x_1 + \ldots + \alpha_r^r x_r)^2 \right] \}$$

$$= F_A^{(r)} \left[ a, \frac{1+\beta_1}{2} - \nu_1, \ldots, \frac{1+\beta_r}{2} - \nu_r; 1 + \frac{1}{2}\beta_1 - \mu_1, \ldots, 1 + \frac{1}{2}\beta_r - \mu_r; \frac{u_1}{\lambda_1^2}, \ldots, \frac{u_r}{\lambda_r^2} \right]$$

where all $\mathrm{Re}(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$, $j = 1,\ldots,r$; $K \neq 0$ and

$$\sum_{i=1}^{r} \left|\frac{u_i}{\lambda_i^2}\right| < 1.$$

$$(6.3.4)\quad M\{F_{1:1;\ldots;1}^{-:2;\ldots;2}\left[\begin{matrix} -:[1+\tfrac{1}{2}\beta_1 \pm \mu_1:1];\ldots;[1+\tfrac{1}{2}\beta_r \pm \mu_r:1]; \\ [c:1,\ldots,1]:[1+\beta_1:2];\ldots;[1+\beta_r:2]; \end{matrix}\right.$$

$$\left. u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2 \right]\}$$

$$= F_B^{(r)}\left[\frac{1+\beta_1}{2}+\nu_1,\ldots,\frac{1+\beta_r}{2}+\nu_r,\ \frac{1+\beta_1}{2}-\nu_1,\ldots,\frac{1+\beta_r}{2}-\nu_r;\ c;\right.$$

$$\left.\frac{u_1}{\lambda_1^2},\ldots,\frac{u_r}{\lambda_r^2}\right]$$

provided that all $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$ and all $\left|\frac{u_j}{\lambda_j^2}\right| < 1$, $j = 1,\ldots,r$, while $K \neq 0$.

$$(6.3.5)\quad M\{F_{1:2;\ldots;2}^{-:2;\ldots;2}\left[\begin{matrix} -:[1+\tfrac{1}{2}\beta_1 \pm \mu_1:1];\ldots \\ [c:1,\ldots,1]:[\frac{\beta_1+1}{2}:1],[\frac{1+\beta_1}{2}+\nu_1:1];\ldots \end{matrix}\right.$$

$$\ldots;[1+\tfrac{1}{2}\beta_r \pm \mu_r:1];$$

$$\ldots;[\frac{\beta_j+1}{2}:1],\ [\frac{\beta_j+1}{2}+\nu_j:1];$$

$$\left. u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2 \right]\}$$

$$= F_B^{(r)}\left[\frac{\beta_1}{2}+1,\ldots,\frac{\beta_r}{2}+1,\frac{\beta_1+1}{2}-\nu_1,\ldots,\frac{\beta_r+1}{2}-\nu_r;c;\right.$$

$$\left.\frac{4u_1}{\lambda_1^2},\ldots,\frac{4u_r}{\lambda_r^2}\right]$$

valid if all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$, $|\frac{u_j}{\lambda_j^2}| < \frac{1}{4}$, $j = 1,\ldots,r$ and $K \neq 0$.

$$(6.3.6)\ M\{F^{2:1;\ldots;1}_{-:3;\ldots;3}\left[\begin{array}{l}[a:1,\ldots,1],[b:1,\ldots,1]: \\ -:[\beta_1+1:2],\ [\frac{1+\beta_1}{2}\pm\nu_1:1];\ldots\end{array}\right.$$

$$[1+\tfrac{1}{2}\beta_1+\mu_1:1];\ldots;[1+\tfrac{1}{2}\beta_r+\mu_r:1];$$

$$;[\beta_r+1:2],\ [\frac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left.u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= F_C^{(r)}\left[a,b,1+\tfrac{1}{2}\beta_1-\mu_1,\ldots,1+\tfrac{1}{2}\beta_r-\mu_r;\frac{u_1}{\lambda_1^2},\ldots,\frac{u_r}{\lambda_r^2}\right],$$

where all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$, $j = 1,\ldots,r$,

$\sum_{i=1}^{r} |(\frac{u_i}{\lambda_i^2})^{1/2}| < 1$, and $K \neq 0$.

$$(6.3.7)\ M\{F^{1:2;\ldots;2}_{1:2;\ldots;2}\left[\begin{array}{l}[a:1,\ldots,1]:[1+\frac{1}{2}\beta_1\pm\mu_1:1];\ldots \\ [c:1,\ldots,1]:[1+\beta_1:2],\ [\frac{1+\beta_1}{2}+\nu_1:1];\ldots\end{array}\right.$$

$$\ldots;\ [1+\tfrac{1}{2}\pm\mu_r:1]\ ;$$

$$\ldots;\ [1+\beta_r:2]\ ,\ [\tfrac{1+\beta_r}{2}+\nu_r:1]\ ;$$

$$\left.u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= F_D^{(r)}\left[a,\ \frac{1+\beta_1}{2}-\nu_1,\ldots,\ \frac{1+\beta_r}{2}-\nu_r;c;\ \frac{u_1}{\lambda_1^2},\ldots,\ \frac{u_r}{\lambda_r^2}\right]$$

provided that all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$, $|\frac{u_j}{\lambda_j^2}| < 1, j = 1,\ldots,r$ and $K \neq 0$.

$$(6.3.8)\ M\{F^{1:2;\ldots;2}_{1:3;\ldots;3}\left[\begin{matrix}[a:1,\ldots,1]\ :\ [1+\frac{1}{2}\beta_1\pm\mu_1:1]\ ;\ldots\\ [c:1,\ldots,1]\ :[\frac{\beta_1+1}{2}:1]\ ,[\frac{1+\beta_1}{2}\pm\nu_1:1]\ ;\ldots\end{matrix}\right.$$

$$;\ [1+\tfrac{1}{2}\beta_r\pm\mu_r:1]\ ;$$

$$;\ [\tfrac{\beta_r+1}{2}:1]\ ,\ [\tfrac{1+\beta_r}{2}\pm\nu_r:1]\ ;$$

$$\left.u_1(\alpha_1'x_1+\ldots+\alpha_r'x_r)^2,\ldots,\ u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= F_D^{(r)}\left[a,1+\frac{\beta_1}{2},\ldots,\ 1+\frac{\beta_r}{2};c;\ \frac{4u_1}{\lambda_1^2},\ldots,\ \frac{4u_r}{\lambda_r^2}\right],$$

where all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$, $|\frac{u_j}{\lambda_j^2}| < \frac{1}{4}$, $j = 1,\ldots,r$ and

$K \neq 0$.

$$(6.3.9)\quad M\left\{F^{-:2;\ldots;2}_{1:2;\ldots;2}\left[\begin{array}{l} -:[1+\tfrac{1}{2}\beta_1 \pm \mu_1:1];\ldots \\ [c:1,\ldots,1] : [\beta_1+1:2],\ [\frac{1+\beta_1}{2}+\nu_1:1];\ldots \end{array}\right.\right.$$

$$;[1+\tfrac{1}{2}\beta_r \pm \mu_r:1];$$

$$;[\beta_r+1:2],\ [\frac{1+\beta_r}{2}+\nu_r:1];$$

$$\left.\left. u_1(\alpha'_1x_1+\ldots+\alpha'_rx_r)^2,\ldots,u_r(\alpha^r_1x_1+\ldots+\alpha^r_rx_r)^2 \right]\right\}$$

$$= \Phi_2^{(r)}\left[\frac{1+\beta_1}{2}-\nu_1,\ldots,\frac{1+\beta_r}{2}-\nu_r;c;\frac{u_1}{\lambda_1^2},\ldots,\frac{u_r}{\lambda_r^2}\right],$$

valid if all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$, $j = 1,\ldots,r$ and $K \neq 0$.

$$(6.3.10)\quad M\left\{F^{-:2;\ldots;2}_{1:3;\ldots;3}\left[\begin{array}{l} -:[1+\tfrac{1}{2}\beta_1 \pm \mu_1:1];\ldots \\ [c:1,\ldots,1] : [\frac{\beta_1+1}{2}:1],[\frac{1+\beta_1}{2}\pm\nu_1:1];\ldots \end{array}\right.\right.$$

$$;[1+\tfrac{1}{2}\beta_r \pm \mu_r:1];$$

$$;[\frac{1+\beta_r}{2}:1],\ [\frac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left.\left. u_1(\alpha'_1x_1+\ldots+\alpha'_rx_r)^2,\ldots,u_r(\alpha^r_1x_1+\ldots+\alpha^r_rx_r)^2 \right]\right\}$$

$$= \Phi_2^{(r)}\left[\frac{\beta_1}{2}+1,\ldots,\frac{\beta_r}{2}+1;c;\frac{4u_1}{\lambda_1^2},\ldots,\frac{4u_r}{\lambda_r^2}\right],$$

where all conditions of (6.3.9) are satisfied.

$$(6.3.11)\quad M\{F^{1:1;\dots;1}_{-:3;\dots;3}\left[\begin{matrix}[a:1,\dots,1] : [1+\frac{\beta_1}{2}+\mu_1:1] ;\dots \\ -:[\beta_1+1:2] , [\frac{1+\beta_1}{2}\pm\nu_j:1] ;\dots\end{matrix}\right.$$

$$; [1+\frac{\beta_r}{2}+\mu_r:1] ;$$

$$;[\beta_r+1:2] , [\frac{1+\beta_r}{2}\pm\nu_r:1] ;$$

$$\left. u_1(\alpha_1'x_1+\dots+\alpha_r'x_r)^2,\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)^2\right]\}$$

$$= \Psi_2^{(r)}\left[a;1+\frac{1}{2}\beta_1-\mu_1,\dots,1+\frac{1}{2}\beta_r-\mu_r; \frac{u_1}{\lambda_1^2},\dots,\frac{u_r}{\lambda_r^2}\right],$$

provided that all $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$, $j = 1,\dots,r$ and $K \neq 0$.

$$(6.3.12)\quad M\{F^{1:2;\dots;2}_{1:2;\dots;3}\left[\begin{matrix}[a:1,\dots,1] : [1+\frac{1}{2}\beta_1\pm\mu_1:1] ;\dots; \\ [c:1,\dots,1] :[1+\beta_1:2]\ [\frac{1+\beta_1}{2}+\nu_1: 1] ;\dots;\end{matrix}\right.$$

$$[1+\frac{1}{2}\beta_r\pm\mu_r:1] ;$$

$$[1+\beta_{r-1}:2],[\frac{1+\beta_{r-1}}{2}+\nu_{r-1}:1];[1+\beta_r:2],[\frac{1+\beta_r}{2}\pm\nu_r:1] ;$$

$$\left. u_1(\alpha_1'x_1+\dots+\alpha_r'x_r)^2,\dots,u_r(\alpha_1^r x_1+\dots+\alpha_r^r x_r)^2\right]\}$$

$$= \Phi_D^{(r)}\left[a,\frac{1+\beta_1}{2}-\nu_1,\dots,\frac{1+\beta_{r-1}}{2}-\nu_{r-1},-;c;\frac{u_1}{\lambda_1^2},\dots,\frac{u_r}{\lambda_r^2}\right],$$

provided that all $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$, $|\frac{u_j}{\lambda_j^2}| < 1$, $j=1,\dots,r$.

$$(6.3.13)\quad M\{F^{1:2;\ldots;2}_{1:3;\ldots;3;4}\left[\begin{matrix}[a:1,\ldots,1]:[1+\tfrac{1}{2}\beta_1\pm\mu_1:1];\ldots;\\ [c:1,\ldots,1]:[\frac{1+\beta_1}{2}:1],[\frac{1+\beta_1}{2}\pm\nu_1:1];\ldots;\end{matrix}\right.$$

$$[1+\tfrac{1}{2}\beta_r\pm\mu_r:1];$$

$$[\frac{1+\beta_{r-1}}{2}:1],[\frac{1+\beta_{r-1}}{2}\pm\nu_{r-1}];[\frac{1+\beta_r}{2}:1],[\frac{\beta_r}{2}+1:1],[\frac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left.u_1(\alpha'_1x_1+\ldots+\alpha'_rx_r)^2,\ldots,u_r(\alpha^r_1x_1+\ldots+\alpha^r_rx_r)^2\right]\}$$

$$=\Phi_D^{(r)}\left[a,1+\frac{\beta_1}{2},\ldots,1+\frac{\beta_{r-1}}{2},-;c;\ \frac{4u_1}{\lambda_1^2},\ldots,\frac{4u_r}{\lambda_r^2}\right],$$

where all $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$, $|\frac{u_j}{\lambda_j^2}| < \frac{1}{4}$, $j = 1,\ldots,r$ and $K \neq 0$.

$$(6.3.14)\quad M\{F^{1:2;\ldots;2}_{1:3;\ldots;3;4}\left[\begin{matrix}[a:1,\ldots,1]:[1+\tfrac{1}{2}\beta_1\pm\mu_1:1];\ldots;\\ [c:1,\ldots,1]:[\frac{\beta_1+1}{2}:1],[\frac{1+\beta_1}{2}\pm\nu_1:1];\ldots;\end{matrix}\right.$$

$$[1+\tfrac{1}{2}\beta_r\pm\mu_r:1];$$

$$[\frac{\beta_{r-1}+1}{2}:1],[\frac{1+\beta_{r-1}}{2}\pm\nu_{r-1}:1],[\frac{\beta_r+1}{2}:1],[\frac{\beta_r}{2}+1:1],[\frac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left.u_1(\alpha'_1x_1+\ldots+\alpha'_rx_r)^2,\ldots,u_r(\alpha^r_1x_1+\ldots+\alpha^r_rx_r)^2\right]\}$$

$$=\Phi_D^{(r)}\left[a,1+\frac{\beta_1}{2},\ldots,1+\frac{\beta_{r-1}}{2},-;c;\ \frac{4u_1}{\lambda_1^2},\ldots,\frac{4u_r}{\lambda_r^2}\right]$$

provided that all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$, $|\frac{u_j}{\lambda_j^2}| < \frac{1}{4}$,

$j = 1, \ldots, r$ and $K \neq 0$.

(6.3.15) $$M\{F^{-:2;\ldots;2}_{1:1;\ldots;1;2}\left[\begin{matrix} -:[1+\frac{1}{2}\beta_1 \pm \mu_1:1];\ldots; \\ [c:1,\ldots,1]:[1+\beta_1:2];\ldots; \end{matrix}\right.$$

$$[1+\frac{1}{2}\beta_r \pm \mu_r:1];$$

$$[1+\beta_{r-1}:2];[1+\beta_r:2],[\frac{1+\beta_r}{2}-\nu_r:1];$$

$$\left. u_1(\alpha_1' x_1+\ldots+\alpha_r' x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= \Xi_1^{(r)}\left[\frac{1+\beta_1}{2}+\nu_1,\ldots,\frac{1+\beta_r}{2}+\nu_r,\frac{1+\beta_1}{2}-\nu_1,\ldots,\frac{1+\beta_{r-1}}{2}\right.$$

$$\left. -\nu_{r-1},-;c;\frac{u_1}{\lambda_1^2},\ldots,\frac{u_r}{\lambda_r^2}\right]$$

valid if all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$ and all $|\frac{u_j}{\lambda_r^2}| < 1$, $j=1,\ldots,r$;

$K \neq 0$.

(6.3.16) $$M\{F^{-:2;\ldots;2}_{1:2;\ldots;2;3}\left[\begin{matrix} -:[1+\frac{1}{2}\beta_1 \pm \mu_1:1];\ldots; \\ [c:1,\ldots,1]:[\beta_1+1:2],[\frac{1+\beta_1}{2}+\nu_1:1]; \end{matrix}\right.$$

$$[1+\frac{1}{2}\beta_r \pm \mu_r:1];$$

$$\ldots;[\beta_{r-1}+1:2],[\frac{1+\beta_{r-1}}{2}+\nu_{r-1}:1];[\beta_r+1:2],[\frac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left. u_1(\alpha_1' x_1+\ldots+\alpha_r' x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= \Phi_3^{(r)} \left[ \frac{1+\beta_1}{2} - \nu_1, \ldots, \frac{1+\beta_{r-1}}{2} - \nu_{r-1}; c; \frac{u_1}{\lambda_1^2}, \ldots, \frac{u_r}{\lambda_r^2} \right]$$

valid if all $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$, $j = 1,\ldots,r$ and $K \neq 0$.

$$(6.3.17)\quad M\{F^{-:2;\ldots;2}_{1:3;\ldots;3;4} \left[ \begin{matrix} -: [1+\frac{1}{2}\beta_1 \pm \mu_1 :1]; \ldots ; \\ [c:1,\ldots,1] : [\frac{\beta_1+1}{2}:1], [\frac{1+\beta_1}{2} \pm \nu_1 :1]; \ldots ; \end{matrix} \right.$$

$$[1+\frac{1}{2}\beta_r \pm \mu_r:1];$$

$$[\frac{1+\beta_{r-1}}{2}:1], [\frac{1+\beta_{r-1}}{2} \pm \nu_{r-1}:1]; [\frac{1+\beta_r}{2}:1], [\frac{1+\beta_{r-1}}{2} \pm \nu_r :1] [\frac{\beta_r}{2}+1:1];$$

$$\left. u_1(\alpha_1' x_1 + \ldots + \alpha_r' x_r)^2, \ldots, u_r(\alpha_1^r x_1 + \ldots + \alpha_r^r x_r)^2 \right] \}$$

$$= \Phi_3^{(r)} \left[ \frac{\beta_1}{2} + 1, \ldots, \frac{\beta_{r-1}}{2} + 1; c; \frac{4u_1}{\lambda_1^2}, \ldots, \frac{4u_r}{\lambda_r^2} \right],$$

where all conditions of (6.3.16) are satisfied.

$$(6.3.18)\quad M\{F^{1:2;\ldots;2}_{2:2;\ldots;2} \left[ \begin{matrix} [a:1,\ldots,1] : [1+\frac{1}{2}\beta_1 \pm \mu_1 :1]; \ldots ; \\ [c:1,\ldots,1,0,\ldots,0], [c':0,\ldots,0,1,\ldots 1] : \end{matrix} \right.$$

$$[1+\frac{1}{2}\beta_r \pm \mu_r :1];$$

$$[1+\beta_1:2], [\frac{1+\beta_1}{2} + \nu_1 :1]; \ldots ; [\frac{1+\beta_r}{2}:2], [\frac{1+\beta_r}{2} + \nu_r :1];$$

$$\left. u_1(\alpha_1' x_1 + \ldots + \alpha_r' x_r)^2, \ldots, u_r(\alpha_1^r x_1 + \ldots + \alpha_r^r x_r)^2 \right] \}$$

$$= {}^{(k)}_{(1)}E_D^{(r)}\left[a,\frac{1+\beta_1}{2}-\nu_1,\ldots,\frac{1+\beta_r}{2}-\nu_r;c;c';\frac{u_1}{\lambda_1^2},\ldots,\frac{u_r}{\lambda_r^2}\right],$$

provided that $K \neq 0$, all $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}|\nu_j| - 1$, $j = 1,\ldots,r$ and $|\frac{u_i}{\lambda_i^2}| < r_i$ with $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_r$, $r_k + r_r = 1$, $r_i$, $i = 1,\ldots,r$ being the associated radii of convergence of the series ${}^{(k)}_{(1)}E_D^{(r)}$.

$$(6.3.19)\quad M\{F^{1:2;\ldots;2}_{2:3;\ldots;3}\left[\begin{matrix}[a:1,\ldots,1]:[1+\frac{1}{2}\beta_1 \pm \mu_1:1];\ldots;\\ [c:1,\ldots,1,0,\ldots,0],[c':0,\ldots,0,1,\ldots,1\end{matrix}\right.$$

$$[1+\frac{1}{2}\beta_r \pm \mu_r:1];$$

$$[\frac{1+\beta_1}{2}:1],[\frac{1+\beta_1}{2}\pm\nu_1:1];\ldots;[\frac{1+\beta_r}{2}:1],[\frac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left.u_1(\alpha_1^1x_1+\ldots+\alpha_r^1x_r)^2,\ldots,u_r(\alpha_1^rx_1+\ldots+\alpha_r^rx_r)^2\right]\}$$

$$= {}^{(k)}_{(1)}E_D^{(r)}\left[a,\frac{1+\beta_1}{2},\ldots,1+\frac{\beta_r}{2};c;c';\frac{4u_1}{\lambda_1^2},\ldots,\frac{4u_r}{\lambda_r^2}\right],$$

where $K \neq 0$, all $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$, $|\frac{u_j}{\lambda_j^2}| < \frac{1}{4}$, $j=1,\ldots,r$ and $|\frac{4u_i}{\lambda_i^2}| < r_i$ with $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_r$, $r_k + r_r = 1$, $r_i$, $i = 1,\ldots,r$

being the associated radii of convergence of the series ${}^{(k)}_{(1)}E_D^{(r)}$

$$(6.3.20)\quad M\{F^{2:2;\ldots;2}_{1:2;\ldots;2}\left[\begin{matrix}[a:1,\ldots,1,0,\ldots,0], \\ [c:1,\ldots,1]:[1+\beta_1:2],[\frac{1+\beta_1}{2}+\nu_1:1];\ldots\end{matrix}\right.$$

$$[a':0,\ldots,0,1,\ldots,1]:[1+\tfrac{1}{2}\beta_1 \pm \mu_1:1];\ldots;[1+\tfrac{1}{2}\beta_r \pm \mu_r:1];$$

$$[1+\beta_r:2],[\frac{1+\beta_r}{2}+\nu_r:1];$$

$$\left.u_1(\alpha_1' x_1+\ldots+\alpha_r' x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= {}^{(k)}_{(2)}E^{(r)}_D\left[a,a',\frac{1+\beta_1}{2}-\nu_1,\ldots,\frac{1+\beta_r}{2}-\nu_r;c;\frac{u_1}{\lambda_1^2},\ldots,\frac{u_r}{\lambda_r^2}\right]$$

provided that all $\mathrm{Re}\,(\beta_j) > 2\,\mathrm{Re}\,|\nu_j| - 1$, $j = 1,\ldots,r$, $K \neq 0$ and $|\frac{u_i}{\lambda_i^2}| < r_i$ with $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_r$, $r_k + r_r = r_k \cdot r_r$ where $r_i$, $i = 1,\ldots,r$ are the associated radii of convergence of ${}^{(k)}_{(2)}E^{(r)}_D$.

$$(6.3.21)\quad M\{F^{2:2;\ldots;2}_{1:3;\ldots;3}\left[\begin{matrix}[a:1,\ldots,1,0,\ldots,0],[a':0,\ldots,0,1,\ldots,1 \\ [c:1,\ldots,1]:[\frac{1+\beta_1}{2}:1],[\frac{1+\beta_1}{2}\pm\nu_1:1];\ldots;\end{matrix}\right.$$

$$[1+\tfrac{1}{2}\beta_1 \pm \mu_1:1];\ldots;[1+\tfrac{1}{2}\beta_r \pm \mu_r:1];$$

$$[\frac{1+\beta_r}{2}:1],[\frac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left.u_1(\alpha_1' x_1+\ldots+\alpha_r' x_r)^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= {}_{(2)}^{(k)}E_D^r\left[a,a',1+\frac{\beta_1}{2},\ldots,1+\frac{\beta_r}{2};c;\frac{4u_1}{\lambda_1^2},\ldots,\frac{4u_r}{\lambda_r^2}\right],$$

provided that all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$, $j = 1,\ldots,r$; $K \neq 0$ and $|\frac{u_i}{\lambda_i^2}| < \frac{r_i}{4}$ with $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_r$, $r_k + r_r = r_k \cdot r_r$ where $r_i$, $i = 1,\ldots,r$ are the associated radii of convergence of the series ${}_{(2)}^{(k)}E_D^{(r)}$.

(6.3.22)
$$M\{F^{3:1;\ldots;1}_{-:3;\ldots;3}\left[\begin{array}{l}[a:1,\ldots,1,0,\ldots,0],[a':0,\ldots,0,1,\ldots,1]\\ -:[1+\beta_1:2],[\frac{1+\beta_1}{2}\pm\nu_1:1];\ldots;\end{array}\right.$$

$$[b:1,\ldots,1]:[1+\tfrac{1}{2}\beta_1+\mu_1:1];[1+\tfrac{1}{2}\beta_r+\mu_r:1];$$

$$[1+\beta_r:2],[\frac{1+\beta_r}{2}\pm\nu_r:1];$$

$$\left.u_r(\alpha_1'x_1+\ldots+\alpha_r')^2,\ldots,u_r(\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^2\right]\}$$

$$= {}_{(1)}^{(k)}E_C^{(r)}\left[a,a'b;1+\tfrac{1}{2}\beta_1-\mu_1,\ldots,1+\tfrac{1}{2}\beta_r-\mu_r;\frac{u_1}{\lambda_1^2},\ldots,\frac{u_r}{\lambda_r^2}\right]$$

provided that all Re $(\beta_j) > 2$ Re $|\nu_j| - 1$, $j = 1,\ldots,r$; $K \neq 0$ and $|\frac{u_i}{\lambda_i^2}| < r_i$ with $(\sqrt{r_1}+\ldots+\sqrt{r_k})^2 + (\sqrt{r_{k+1}}+\ldots+\sqrt{r_r})^2 = 1$, $r_i$, $i = 1,\ldots,r$ being the associated radii of convergence of the series ${}_{(1)}^{(k)}E_C^{(r)}$.

Applying the same techniques and specialising the number of variables, we can obtain multiple Whittaker transforms of hypergeometric functions of different variables.

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CHAPTER VII

# MULTIDIMENSIONAL GAUSS' TRANSFORMS OF MULTIPLE HYPERGEOMETRIC FUNCTIONS IN SEVERAL VARIABLES

7.1. Introduction. In previous chapters V and VI, we have studied multidimensional Whittaker transforms of multiple hypergeometric functions in several variables. Making an appeal to fractional integration, Chandel [1,2] has also obtained the Eulerion integral representations of multiple hypergeometric functions of several variables by using the operator

$$(7.1.1)\quad \Omega\{\ \} = \prod_{j=1}^{n} \frac{\Gamma(\nu_j)}{\Gamma(\beta_j)\,\Gamma(\nu_j-\beta_j)} \int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{\beta_i-1}(1-t_i)^{\nu_i-\beta_i-1}$$

$$\{\ \}\, dt_1 \ldots dt_n,$$

where $0 < \mathrm{Re}\,(\beta_i) < \mathrm{Re}\,(\nu_i)$, $i = 1,\ldots,n$.

Further in the same papers he has also given the extensions of the above work by using the operator

$$(7.1.2)\quad R\{\ \} = \prod_{j=1}^{r} \frac{\Gamma(\nu_j)\ \Gamma(\lambda_j)}{\Gamma(\beta_j)\Gamma(\mu_j)\Gamma(\nu_j+\lambda_j-\beta_j-\mu_j)}$$

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} t_i^{\mu_i-1}\ (1-t_i)^{\nu_i+\lambda_i-\mu_i-\beta_i-1}$$

$${}_2F_1\ [\nu_i-\beta_i, \lambda_i-\beta_i;\ \nu_i+\lambda_i-\mu_i-\beta_i;\ 1-t_i]\ \{\quad\}\ dt_1\ \ldots\ dt_n,$$

where $0 < \mathrm{Re}\,(\mu_j) < \mathrm{Re}\,(\nu_j + \lambda_j - \beta_j)$, $0 < \mathrm{Re}(\beta_j) < \mathrm{Re}(\nu_j)$, $j = 1,\ldots,n$.

In this chapter, we further extend the above work through an easy approach without using fractional integration, by introducing the following two multidimensional Gauss' transforms

(7.1.3) $A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{\quad\}$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_r)\ \Gamma(\sigma)\ \Gamma(\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_r)\ \Gamma(\delta)\ \Gamma(\beta+\sigma-\delta)\ \Gamma(\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta)}$$

$$\int_0^\infty\ldots\int_0^\infty x_1^{\alpha_1-1}\ldots x_r^{\alpha_r-1}\ (x_1+\ldots+\alpha_r)^{\gamma}\,(1+x_1+\ldots+x_r)^{-\sigma}$$

$${}_2F_1\ [\beta,\gamma;\delta;-(x_1+\ldots+x_r)]\ \{\quad\}\ dx_1\ \ldots\ dx_r$$

where $\mathrm{Re}(\delta) > 0$, $\mathrm{Re}(\beta+\sigma-\delta) > 0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > 0$ and $\mathrm{Re}\,(\alpha_j) > 0$, $j = 1,\ldots,r$

(7.1.4) $$B\{\quad\} = \prod_{i=1}^{r} \frac{K\ \Gamma(\sigma_i)\ \Gamma(\beta_i+\gamma_i+\sigma_i-\delta_i)}{\Gamma(\delta_i)\ \Gamma(\beta_i+\sigma_i-\delta_i)\ \Gamma(\gamma_i+\sigma_i-\delta_i)}$$

$$\int_0^\infty\ldots\int_0^\infty \prod_{i=1}^{r}(\alpha_1^i x_1+\ldots+\alpha_r^i x_i)^{\gamma_i-1}\ (1+\alpha_1^i x_1+\ldots+\alpha_r^i x_r)^{-\sigma_i}$$

$${}_2F_1\ [\beta_i,\gamma_i;\delta_i;-(\alpha_1^i x_1+\ldots+\alpha_r^i x_r)]\ \{\quad\}\ dx_1\ \ldots\ dx_r$$

provided that all $\mathrm{Re}(\delta_i) > 0$, $\mathrm{Re}\,(\beta_i+\sigma_i-\delta_i) > 0$, $\mathrm{Re}(\gamma_i+\sigma_i-\delta_i) > 0$

$$i = 1,\ldots,r \text{ and } K = \begin{vmatrix} \alpha'_1 & \alpha'_2 & \ldots & \alpha'_r \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_r^2 \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^r & \alpha_2^r & \ldots & \alpha_r^r \end{vmatrix} \neq 0,$$

to obtain multidimensional integral representations of multiple hypergeometric function of Srivastava and Daoust [5]. Their special cases will also be discussed to establish multidimensional Gauss' transforms of multiple hypergeometric functions of several variables defined by Chandel [2], Exton [3] and Lauricella [4]. These transforms have been named Gauss' transforms' due to presence of Gauss' hypergeometric series ${}_2F_1$.

## 7.2. Transform $A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}$

Consider

$$\int_0^\infty \cdots \int_0^\infty x_1^{\alpha_1-1} \cdots x_r^{\alpha_r-1} (x_1+\ldots+x_r)^\gamma (1+x_1+\ldots+x_r)^{-\sigma}$$

$${}_2F_1\,[\beta,\gamma;\delta;-(x_1+\ldots+x_r)]dx_1 \ldots dx_r$$

$$= \frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_r)}{\Gamma(\alpha_1+\ldots+\alpha_r)} \int_0^\infty t^{\gamma+\alpha_1+\ldots+\alpha_r-1} (1+t)^{-\sigma}\, {}_2F_1\,[\beta,\gamma;\delta;-t]\,dt$$

$$= \frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_r)\Gamma(\delta)\,\Gamma(\beta+\sigma-\delta)\,\Gamma(\alpha_1+\ldots+\alpha_r+\sigma-\delta)}{\Gamma(\alpha_1+\ldots+\alpha_r)\,\Gamma(\sigma)\,\Gamma(\beta+\gamma+\sigma+\alpha_1+\ldots+\alpha_r-\delta)},$$

where $\mathrm{Re}(\delta) > 0$, $\mathrm{Re}(\beta+\sigma-\delta) > 0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > 0$, $\mathrm{Re}(\alpha_j) > 0$, $j = 1,\ldots,r$.

For brevity, this suggests to introduce the multidimensional Gauss' transform defined by (7.1.3).

It is easy to prove that

(7.2.1) $A_{\beta,\gamma,\delta,\sigma}^{\alpha_1,\ldots,\alpha_r}\{1\} = 1$

(7.2.2) $A_{\beta,\gamma,\delta,\sigma}^{\alpha_1,\ldots,\alpha_r}\{x_1^{m_1}\ldots x_r^{m_r}\}$

$$= \frac{(\alpha_1)_{m_1}\ldots(\alpha_r)_{m_r}(\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta)_{m_1+\ldots+m_r}}{(\beta+\gamma+\sigma+\alpha_1+\ldots+\alpha_r-\delta)_{m_1+\ldots+m_r}}$$

and

(7.2.3) $A_{\beta,\gamma,\delta,\sigma}^{\alpha_1,\ldots,\alpha_r}\{x_1^{m_1\xi_1}\ldots x_r^{m_r\xi_r}(1+x_1+\ldots+x_r)^{-(m_1\eta_1+\ldots+m_r\eta_r)}\}$

$$= \frac{(\alpha_1)_{m_1\xi_1}\ldots(\alpha_r)_{m_r\xi_r}(\beta+\sigma-\delta)_{m_1\eta_1+\ldots+m_r\eta_r}}{(\alpha_1+\ldots+\alpha_r)_{m_1\xi_1+\ldots+m_r\xi_r}(\sigma)_{m_1\eta_1+\ldots+m_r\eta_r}}$$

$$\frac{(\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta)_{m_1(\xi_1+\eta_1)+\ldots+m_r(\xi_r+\eta_r)}}{(\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta)_{m_1(\xi_1+\eta_1)+\ldots+m_r(\xi_r+\eta_r)}}$$

Now an appeal to (7.2.2) and (7.2.3) respectively gives the following two operational results involving multiple hypergeometric function of Srivastava and Dauost [5] :

$$(7.2.4)\quad A_{\beta,\gamma,\delta,\sigma}^{\alpha_1,\ldots,\alpha_r}\{F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left[\begin{array}{l}[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\Psi',\ldots,\Psi^{(r)}]:\end{array}\right.$$

$$\left.\begin{array}{l}[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];\\ \\ [(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}];\end{array} u_1x_1^{\xi_1},\ldots,u_rx_r^{\xi_r}\right]\}$$

$$= F^{A+1:B'+1;\ldots;B^{(r)}+1}_{C+2:D';\ldots;D^{(r)}}\left[\begin{array}{l}[(a):\theta',\ldots,\theta^{(r)}],\\ [(c):\Psi',\ldots,\Psi^{(r)}],[\alpha_1+\ldots+\alpha_r:\xi_1,\ldots,\xi_r],\end{array}\right.$$

$$\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:\xi_1,\ldots,\xi_r]:[(b'):\Phi'],[\alpha_1:\xi_1];[(b^{(r)}):\Phi^{(r)}],[\alpha_r:\xi_r]:$$

$$[\beta+\sigma+\gamma+\alpha_1+\ldots+\alpha_r-\delta:\xi_1,\ldots,\xi_r]:[(d'):\delta'];\ldots;[(d^{(r)}(:\delta^{(r)}];$$

$$\left.u_1,\ldots,u_r\right]$$

provided that $Re(\delta) > 0$, $Re(\beta+\sigma-\delta) > 0$, $Re(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > 0$,

$Re(\alpha_i) > 0$, and $1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)} > 0$,

$i = 1,\ldots,r.$

$$(7.2.5)\quad A_{\beta,\gamma,\delta,\sigma}^{\alpha_1,\ldots,\alpha_r}\{F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left[\begin{array}{l}[(a):\theta',\ldots,\theta^{(r)}]:\\ [(c):\Psi',\ldots,\Psi^{(r)}]:\end{array}\right.$$

$$[(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}] ;$$

$$[(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}] ;$$

$$\left. u_1 x_1^{\xi_1}(1+x_1+\ldots+x_r)^{-\eta_1},\ldots,u_r x_r^{\xi_r}(1+x_1+\ldots+x_r)^{-\eta_r} \right] \}$$

$$= F^{A+2:B'+1;\ldots;B^{(r)}+1}_{C+3:D';\ldots;D^{(r)}} \left[ \begin{array}{l} [(a):\theta',\ldots,\theta^{(r)}],[\beta+\sigma-\delta:\eta_1,\ldots,\eta_r], \\ [(c):\Psi',\ldots,\Psi^{(r)}],[\alpha_1+\ldots+\alpha_r:\xi_1,\ldots,\xi_r], \end{array} \right.$$

$$[\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:\xi_1+\eta_1,\ldots,\xi_r+\eta_r]:[(b'):\phi'],[\alpha_1:\xi_1];\ldots;$$

$$[\sigma:\eta_1,\ldots,\eta_r],[\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta:\xi_1+\eta_1,\ldots,\xi_r+\eta_r]:[(d'):\delta'];\ldots;$$

$$\left. \begin{array}{ll} [(b^{(r)}):\phi^{(r)}],[\alpha_r:\xi_r], & \\ & u_1,\ldots,u_r \\ [(d^{(r)}):\delta^{(r)}] ; & \end{array} \right]$$

provide that all conditions of (7.2.4) are satisfied.

7.3 <u>Special case of the Operator</u>. For $\delta = \gamma + \sigma$, we have

$$(7.3.1) \quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{x_1^{m_1},\ldots,x_r^{m_r}\} = \frac{(\alpha_1)_{m_1}\ldots(\alpha_r)_{m_r}}{(\alpha_1+\ldots+\alpha_r+\beta)_{m_1+\ldots+m_r}}$$

Therefore, we obtain

$$(7.3.2)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{F^{(r)}_A(a,b_1,\ldots,b_r;\alpha_1,\ldots,\alpha_r;u_1x_1,\ldots,u_rx_r)\}$$

$$= F^{(r)}_D\ (a,b_1,\ldots,b_r;\alpha_1+\ldots+\alpha_r+\beta;\ u_1,\ldots,u_r)$$

provided that Re $(\gamma+\sigma) > 0$, Re $(\beta-\gamma) > 0$, all Re $(\alpha_j) > 0$, $j = 1,\ldots,r$ and $\sum_{j=1}^{r} |u_j| < 1$.

$$(7.3.3)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{F^{(r)}_A(\alpha_1+\ldots+\alpha_r+\beta,b_1,\ldots,b_r;c_1,\ldots,c_r;$$

$$u_1x_1,\ldots,u_rx_r)\}$$

$$= \prod_{j=1}^{r}\ {}_2F_1\ (\alpha_j,b_j;c_j;u_j),$$

which is true if Re $(\gamma+\sigma) > 0$, Re $(\beta-\gamma) > 0$, all Re $(\alpha_j) > 0$, $|u_j| < 1$, $j = 1,\ldots,r$.

$$(7.3.4)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{F^{(r)}_C(\alpha_1+\ldots+\alpha_r+\beta,b;c_1,\ldots,c_r;u_1x_1,\ldots,u_rx_r)\}$$

$$= F^{(r)}_A\ (b,\alpha_1,\ldots,\alpha_r;c_1,\ldots,c_r;u_1,\ldots,u_r)$$

valid if Re $(\gamma+\sigma) > 0$, Re $(\beta-\gamma) > 0$, $\sum_{j=1}^{r} |u_j| < 1$, Re $(\alpha_j) > 0$, $j = 1,\ldots,r$.

$$(7.3.5)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{F^{(r)}_D(\alpha_1+\ldots+\alpha_r+\beta,b_1,\ldots,b_r;c;u_1x_1,\ldots,u_rx_r)\}$$

$$= F^{(r)}_B\ (\alpha_1,\ldots,\alpha_r,b_1,\ldots,b_r;c;\ u_1,\ldots,u_r),$$

where Re $(\gamma+\sigma) > 0$ Re $(\beta-\gamma) > 0$, all $(\alpha_j) > 0$, $|u_j| < 1$, $j = 1,\ldots,r$.

$$(7.3.6)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{F_A^{(r)}(a,b_1,\ldots,b_r;\alpha_1,\ldots,\alpha_r;u_1x_1,\ldots,u_rx_r)\}$$

$$= F_D^{(r)}\ (a,b_1,\ldots,b_r;\beta+\alpha_1+\ldots+\alpha_r;\ u_1,\ldots,u_r),$$

provided that Re $(\gamma+\sigma) > 0$, Re $(\beta-\gamma) > 0$, Re $(\alpha_j) > 0$ and $|u_j| < 1$, $j = 1,\ldots,r$.

$$(7.3.7)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{F_C^{(r)}\ (a,b;\alpha_1,\ldots,\alpha_r;u_1x_1,\ldots,u_rx_r)\}$$

$$= {}_2F_1\ (a,b;\alpha_1+\ldots+\alpha_r+\beta;\ u_1+\ldots+u_r),$$

valid if Re $(\gamma+\sigma) > 0$, Re $(\beta-\gamma) > 0$, all Re $(\alpha_j) > 0$, $j=1,\ldots,r$

and $|\sum_{j=1}^{r} u_j| < 1$

$$(7.3.8)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{\Psi_2^{(r)}(\beta,\alpha_1+\ldots+\alpha_r;c_1,\ldots,c_r;u_1x_1,\ldots,u_rx_r)\}$$

$$= \prod_{j=1}^{r}\ {}_1F_1\ (\alpha_j;c_j;u_j),$$

where Re $(\gamma+\sigma) > 0$, Re $(\beta-\gamma) > 0$, Re $(\alpha_j) > 0$, $|u_j| < 1$, $j = 1,\ldots,r$.

$$(7.3.9)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{\Psi_2^{(r)}\ (a;\alpha_1,\ldots,\alpha_r;u_1x_1,\ldots,u_rx_r)\}$$

$$= {}_1F_1\ (a;\alpha_1+\ldots+\alpha_r+\beta;\ u_1+\ldots+u_r)$$

provided that $\mathrm{Re}\,(\gamma+\sigma) > 0$, $\mathrm{Re}\,(\beta-\gamma) > 0$, all $\mathrm{Re}\,(\alpha_j) > 0$, $j = 1,\ldots,r$.

$$(7.3.10)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{\Psi_2^{(r)}(-n;\alpha_1,\ldots,\alpha_r;u_1x_1,\ldots,u_rx_r)\}$$

$$= \frac{n!}{(\alpha_1+\ldots+\alpha_r+\beta)_n} L_n^{(\alpha_1+\ldots+\alpha_r+\beta-1)}(u_1+\ldots+u_r)$$

valid if $\mathrm{Re}\,(\gamma+\sigma) > 0$, $\mathrm{Re}\,(\beta-\gamma) > 0$, $\mathrm{Re}\,(\alpha_j) > 0$, $j = 1,\ldots,r$.

$$(7.3.11)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{\Phi_D^{(r)}(\alpha_1+\ldots+\alpha_r+\beta, b_1,\ldots,b_{r-1},-;c;$$

$$u_1x_1,\ldots,u_rx_r)\}$$

$$= \Xi_1^{(r)}(\alpha_1,\ldots,\alpha_r, b_1,\ldots,b_{r-1},-;c;u_1,\ldots,u_r)$$

where $\mathrm{Re}\,(\gamma+\sigma) > 0$, $\mathrm{Re}\,(\beta-\gamma) > 0$, $\mathrm{Re}\,(\alpha_j) > 0$, $|u_j| < 1$, $j = 1,\ldots,r$.

$$(7.3.12)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{{}^{(k)}_{(1)}E_D^{(r)}(\alpha_1+\ldots+\alpha_r+\beta,\beta_1,\ldots,\beta_r;\gamma,\gamma';$$

$$u_1x_1,\ldots,u_rx_r)\}$$

$$= F_B^{(k)}(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k;\gamma;u_1,\ldots,u_k)$$

$$F_B^{(r-k)}(\alpha_{k+1},\ldots,\alpha_r,\beta_{k+1},\ldots,\beta_r;\gamma';u_{k+1},\ldots,u_r),$$

valid if Re $(\gamma+\sigma) > 0$, Re $(\beta-\gamma) > 0$, Re $(\alpha_j) > 0$ and $|u_j| < 1$, $j = 1,\ldots,r$.

$$(7.3.13)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\gamma+\sigma,\sigma}\{{}^{(k)}_{(1)}E^{(r)}_C(\delta,\delta',\alpha_1+\ldots+\alpha_r+\beta;\gamma_1,\ldots,\gamma_r; u_1x_1,\ldots,u_rx_r)\}$$

$$= F^{(k)}_A(\delta,\alpha_1,\ldots,\alpha_k;\gamma_1,\ldots,\gamma_k;\ u_1,\ldots,u_k)$$

$$F^{(r-k)}_A(\delta',\alpha_{k+1},\ldots,\alpha_r;\ \gamma_{k+1},\ldots,\gamma_r;\ u_{k+1},\ldots,u_r)$$

where Re $(\gamma+\sigma) > 0$, Re $(\beta-\gamma) > 0$, all Re $(\alpha_j) > 0$, $j = 1,\ldots,r$ and $\sum_{j=1}^{k} |u_j| < 1$ and $\sum_{j=k+1}^{r} |u_j| < 1$.

## 7.4 Some special cases of parameters.

Specialising the parameters in (7.2.4), we obtain

$$(7.4.1)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{2:-;\ldots;-;}_{-:1;\ldots;1;}\left[\begin{matrix}[\alpha_1+\ldots+\alpha_r:1,\ldots,1]\ ,\ [\beta+\gamma+\sigma+\alpha_1+\ldots+\alpha_r-\delta:1,\ldots,1]\ :-;\ldots;-; \\ -:\ [c_1:1]\ ;\ldots;\ [c_r:1]\ ;\end{matrix}\ u_1x_1,\ldots,u_rx_r\right]\}$$

$$= F^{(r)}_A(\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta,\alpha_1,\ldots,\alpha_r;c_1,\ldots,c_r;u_1,\ldots,u_r)$$

provided Re $(\delta) > 0$, Re $(\beta+\sigma-\delta) > 0$, Re $(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > 0$,

all Re $(\alpha_j) > 0$, $j = 1,\ldots,r$ and $\sum_{j=1}^{r} |u_j| < 1$.

$$(7.4.2)\quad A_{\beta,\gamma,\delta,\sigma}^{\alpha_1,\ldots,\alpha_r}\{F_{4:-;\ldots;-}^{1:1;\ldots;1}\left[\begin{matrix}[\beta+\gamma+\sigma+\alpha_1+\ldots+\alpha_r-\delta:1,\ldots,1]:\\ [\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:1,\ldots,1];\end{matrix}\right.$$

$$\left.\begin{matrix}[\beta_1:1];\ldots;[\beta_r:1];\\ -;\ldots;-;\end{matrix}\quad u_1x_1,\ldots,u_rx_r\right]\}$$

$$= F_B^{(r)}(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_r;\alpha_1+\ldots+\alpha_r;u_1,\ldots,u_r)$$

valid if Re $(\delta) > 0$, Re $(\beta+\sigma-\delta) > 0$, Re$(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > 0$,

all Re $(\alpha_j) > 0$, and $|u_j| < 1$, $j = 1,\ldots,r$.

$$(7.4.3)\quad A_{\beta,\gamma,\delta,\sigma}^{\alpha_1,\ldots,\alpha_r}\{F_{1:-;\ldots;-}^{1:1;\ldots;1}\left[\begin{matrix}[\alpha_1+\ldots+\alpha_r:1,\ldots,1]:\\ [\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:1,\ldots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[\beta_1:1];\ldots;[\beta_r:1];\\ -;\ldots;-;\end{matrix}\quad u_1x_1,\ldots,u_rx_r\right]\}$$

$$= F_B^{(r)}(\alpha_1,\ldots,\alpha_r;\beta_1,\ldots,\beta_r;\beta+\gamma+\sigma+\alpha_1+\ldots+\alpha_r-\delta;\ u_1,\ldots,u_r)$$

provided all conditions of (7.4.2) are satisfied.

$$(7.4.4)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{3:-;\ldots;-}_{-:2;\ldots;2}\left[\begin{matrix}[\alpha_1+\ldots+\alpha_r:1,\ldots,1],\\ -:[\alpha_1:1],[c_1:1];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[\beta+\gamma+\sigma+\alpha_1+\ldots+\alpha_r-\delta:1,\ldots,1],[a:1,\ldots,1]:-;\ldots;-;\\ [\alpha_r:1],[c_r:1];\end{matrix}\quad u_1x_1,\ldots,u_rx_r\right]\}$$

$$= F_C^{(r)}(a,\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta;\ c_1,\ldots,c_r;\ u_1,\ldots,u_r)$$

provided that $\mathrm{Re}\,(\delta) > 0$, $\mathrm{Re}\,(\beta+\sigma-\delta) > 0$, $\mathrm{Re}\,(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > ($

all $\mathrm{Re}\,(\alpha_j) > 0$, $j = 1,\ldots,r$ and $\sum_{j=1}^{r} |(u_j)^{1/2}| < 1.$

$$(7.4.5)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{[1-(u_1x_1+\ldots+u_rx_r)]^{-(\alpha_1+\ldots+\alpha_r)}\}$$

$$= F_D^{(r)}(\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta,\alpha_1,\ldots,\alpha_r;\beta+\gamma+\sigma+\alpha_1+\ldots+\alpha_r-\delta;u_1,\ldots,u_r),$$

valid if $\mathrm{Re}\,(\delta) > 0$, $\mathrm{Re}\,(\beta+\sigma-\delta) > 0$, $\mathrm{Re}\,(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > 0$, all $\mathrm{Re}\,(\alpha_j) > 0$ and $|u_j| < 1$, $j = 1,\ldots,r.$

$$(7.4.6)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{[1-(u_1x_1+\ldots+u_rx_r)]^{-(\beta+\gamma+\sigma+\alpha_1+\ldots+\alpha_r-\delta)}\}$$

$$= F_D^{(r)}(\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta,\ \alpha_1,\ldots,\alpha_r;\alpha_1+\ldots+\alpha_r;u_1,\ldots,u_r)$$

where all the conditions of (7.4.5) are satisfied.

$$(7.4.7)\quad A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{{}_1F_1[\beta+\gamma+\sigma+\alpha_1+\dots+\alpha_r-\delta;\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta;$$

$$u_1x_1+\dots+u_rx_r]\}$$

$$=\Phi_2^{(r)}(\alpha_1,\dots,\alpha_r;\ \alpha_1+\dots+\alpha_r;\ u_1,\dots,u_r),$$

where $Re(\delta)>0$, $Re(\beta+\sigma-\delta)>0$, $Re(\alpha_1+\dots+\alpha_r+\sigma+\gamma-\delta)>0$, and all $Re(\alpha_j)>0$, $j=1,\dots,r$.

$$(7.4.8)\quad A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{{}_1F_1[\alpha_1+\dots+\alpha_r;\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta;u_1x_1+\dots+u_rx_r]$$

$$=\Phi_2^{(r)}(\alpha_1,\dots,\alpha_r;\ \alpha_1+\dots+\alpha_r+\beta+\gamma+\sigma-\delta;\ u_1,\dots,u_r)$$

where all conditions of (7.4.2) are satisfied.

$$(7.4.9)\quad A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{2:-;\dots;-}_{-:2;\dots;2}\left[\begin{matrix}[\alpha_1+\dots+\alpha_r:1,\dots,1],\\ -:[\alpha_1:1],[\beta_1:1];\dots;\end{matrix}\right.$$

$$\left.\begin{matrix}[\beta+\gamma+\sigma+\alpha_1+\dots+\alpha_r-\delta:1,\dots,1]:-;\dots;-;\\ [\alpha_r:1],[\beta_r:1];\end{matrix}\quad u_1x_1,\dots,u_rx_r\right]\}$$

$$=\Psi_2^{(r)}(\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta;\ \beta_1,\dots,\beta_r;\ u_1,\dots,u_r)$$

provided that all the conditions of (7.4.7) are satisfied.

$$(7.4.10)\quad A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{1:1;\dots;1;-}_{1:-;\dots;-}\left[\begin{matrix}[\beta+\gamma+\sigma+\alpha_1+\dots+\alpha_r-\delta:1,\dots,1]:\\ [\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta:1,\dots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix} [\beta_1:1];\dots;[\beta_{r-1}:1]\ ;-; \\ -;\dots;-; \end{matrix}\ u_1x_1,\dots,u_rx_r\right]\Big\}$$

$$= \Xi_1^{(r)}(\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_{r-1};\ \alpha_1+\dots+\alpha_r;u_1,\dots,u_r)$$

valid if Re $(\delta) > 0$, Re $(\beta+\sigma-\delta) > 0$, Re $(\alpha_1+\dots+\alpha_r+\sigma+\gamma-\delta) > 0$, all Re $(\alpha_j) > 0$ and $|u_j| < 1$, $j = 1,\dots,r$.

$$(7.4.11)\quad A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\Big\{F^{1:1;\dots;1;-}_{1:-;\dots;-}\left[\begin{matrix}[\alpha_1+\dots+\alpha_r:1,\dots,1]: \\ [\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta:1,\dots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix} [\beta_1:1]\ ;\ [\beta_{r-1}:1];-; \\ -;\dots;-; \end{matrix}\ u_1x_1,\dots,u_rx_r\right]\Big\}$$

$$= \Xi_1^{(r)}(\alpha_1,\dots,\alpha_r,\beta_1,\dots,\beta_{r-1};\alpha_1+\dots+\alpha_r+\beta+\gamma+\sigma-\delta;\ u_1,\dots,u_r)$$

provided that all conditions of (7.4.10) are satisfied.

$$(7.4.12)\quad A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\Big\{F^{4:-;\dots;-}_{-:2;\dots;2}\left[\begin{matrix}[\alpha_1+\dots+\alpha_r:1,\dots,1]\ , \\ -:[\alpha_1:1]\ ,\end{matrix}\right.$$

$$[\alpha_1+\dots+\alpha_r+\beta+\gamma+\sigma-\delta:1,\dots,1],[a:1,\dots,1,0,\dots,0],$$

$$\left.\begin{matrix} [c_1:1];\dots; \\ [a':0,\dots,0,1,\dots,1]\ :-;-;\dots;-; \\ [\alpha_r:1]\ ,\ [c_r:1]\ ; \end{matrix}\ u_1x_1,\dots,u_rx_r\right]\Big\}$$

$$= {}^{(k)}_{(1)}E^{(r)}_{C}\ (a, a', \alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta; c_1, \dots, c_r; u_1, \dots, u_r)$$

where Re $(\delta) > 0$, Re $(\beta+\sigma-\delta) > 0$, Re $(\alpha_1+\dots+\alpha_r+\sigma+\gamma-\delta) > 0$, all Re $(\alpha_j) > 0$, $j = 1, \dots, r$ and $|u_i| < r_i$ with $(\sqrt{r_1}+\dots+\sqrt{r_k})^2 + (\sqrt{r_{k+1}}+\dots+\sqrt{r_r})^2 = 1$, $r_i$, $i = 1, \dots, r$ being the associated radii of convergence of the series ${}^{(k)}_{(1)}E^{(r)}_{C}$

$$(7.4.13)\quad A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\ \{F^{2:-;\dots;-}_{2:-;\dots;-}\left[\begin{matrix}[\alpha_1+\dots+\alpha_r:1,\dots,1], \\ [a:1,\dots,1,0,\dots,0],\end{matrix}\right.$$

$$\left.\begin{matrix}[\alpha_1+\dots+\alpha_r+\beta+\gamma+\sigma-\delta:1,\dots,1]:-;\dots;-; \\ [a':0,\dots,0,1,\dots,1]:-;\dots;-;\end{matrix}\quad u_1x_1,\dots,u_rx_r\right]\}$$

$$= {}^{(k)}_{(1)}E^{(r)}_{D}\ (\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta, \alpha_1, \dots, \alpha_r; a, a'; u_1, \dots, u_r)$$

where Re $\delta > 0$, Re $(\beta+\sigma-\delta) > 0$, Re $(\alpha_1+\dots+\alpha_r+\sigma+\gamma-\delta) > 0$, all Re $(\alpha_j) > 0$, $j = 1, \dots, r$ and $|u_i| < r_i$ with $r_1 = \dots = r_k$, $r_{k+1} = \dots = r_r$, $r_k+r_r = 1$, $i = 1, \dots, r$ being the associated radii of convergence of the series ${}^{(k)}_{(1)}E^{(r)}_{D}$.

$$(7.4.14)\quad A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\ \{F^{3:-;\dots;-}_{1:-;\dots;-}\left[\begin{matrix}[a:1,\dots,1,0,\dots,0], \\ [\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta:1,\dots,1]:\end{matrix}\right.$$

$$\left.\begin{matrix}[a':0,\dots,0,1,\dots,1]\ ,-;\dots;-; \\ -;\dots;-;\end{matrix}\quad u_1x_1,\dots,u_rx_r\right]\}$$

$$= {}^{(k)}_{(2)}E_D^{(r)} \ (a,a',\alpha_1,\ldots,\alpha_r;\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta \ ; \ u_1,\ldots,u_r)$$

provided that $\mathrm{Re}\ (\delta) > 0$, $\mathrm{Re}\ (\beta+\sigma-\delta) > 0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > 0$, all $\mathrm{Re}\ (\alpha_j) > 0$ and if $|u_i| < r_i$, with $r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_r$, $r_k+r_r = r_k \cdot r_r$ where $r_i$, $i = 1,\ldots,r$ are the asociated radii of convergence of ${}^{(k)}_{(2)}E_D^{(r)}$.

$$(7.4.15) \quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma} \{F^{3:-;\ldots;-}_{1:-;\ldots;-}\left[\begin{matrix}[a:1,\ldots,1,0,\ldots,0], \\ [\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:1,\ldots1] :\end{matrix}\right.$$

$$[a':0,\ldots,0,1,\ldots,1],[\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta:1,\ldots,1]:-;\ldots;-;$$

$$-;\ldots;-;$$

$$\left. u_1x_1,\ldots,u_rx_r \right]\}$$

$$= {}^{(k)}_{(2)}E_D^{(r)} \ (a,a',\alpha_1,\ldots,\alpha_r;\alpha_1+\ldots+\alpha_r; \ u_1,\ldots,u_r)$$

provided that all the conditions of (7.4.14) are satisfied.

Now specialising the parameters in (7.2.5), we establish the following results :

$$(7.4.16) \quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma} \{{}_3F_3\left[\begin{matrix}\sigma, \dfrac{\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta}{2}, \\ \beta+\sigma-\delta, \dfrac{\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta}{2},\end{matrix}\right.$$

$$\left.\begin{matrix} \dfrac{1+\alpha_1+\cdots+\alpha_r+\beta+\gamma+\sigma-\delta}{2}\ ; \\ \dfrac{1+\alpha_1+\cdots+\alpha_r+\gamma+\sigma-\delta}{2}\ ; \end{matrix}\ \dfrac{u_1x_1+\cdots+u_rx_r}{1+x_1+\cdots+x_r}\right]\Big\}$$

$$= \Phi_2^{(r)}\ (\alpha_1,\ldots,\alpha_r;\ \alpha_1+\cdots+\alpha_r;\ u_1,\ldots,u_r),$$

where $\mathrm{Re}\ (\delta) > 0$, $\mathrm{Re}\ (\beta+\sigma-\delta) > 0$, $\mathrm{Re}(\alpha_1+\cdots+\alpha_r+\sigma+\gamma-\delta) > 0$ and $\mathrm{Re}\ (\alpha_j) > 0$, $j = 1,\ldots,r$.

$$(7.4.17)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\ \Big\{{}_3F_3\left[\begin{matrix} \alpha_1+\cdots+\alpha_r, & \dfrac{\beta+\gamma+\sigma-\delta+\alpha_1+\cdots+\alpha_r}{2}, \\ \beta-\sigma-\delta\ , & \dfrac{\alpha_1+\cdots+\alpha_r+\gamma+\sigma-\delta}{2}, \end{matrix}\right.$$

$$\left.\begin{matrix} \dfrac{1+\alpha_1+\cdots+\alpha_r+\beta+\gamma+\sigma-\delta}{2}\ ; \\ \dfrac{1+\alpha_1+\cdots+\alpha_r+\gamma+\sigma-\delta}{2}\ ; \end{matrix}\ \dfrac{u_1x_1+\cdots+u_rx_r}{1+x_1+\cdots+x_r}\right]\Big\}$$

$$= \Phi_2^{(r)}\ (\alpha_1,\ldots,\alpha_r;\ \sigma;\ u_1,\ldots,u_r)$$

valid if all the conditions of (7.4.16) are satisfied.

$$(7.4.18)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\ \Big\{{}_3F_3\left[\begin{matrix} \sigma,\alpha_1+\cdots+\alpha_r, & \\ \beta+\sigma-\delta\ , & \dfrac{\alpha_1+\cdots+\alpha_r+\gamma+\sigma-\delta}{2}, \end{matrix}\right.$$

$$\left.\begin{matrix} \dfrac{1+\beta+\gamma+\sigma-\delta+\alpha_1+\cdots+\alpha_r}{2}\ ; \\ \dfrac{\alpha_1+\cdots+\alpha_r+\gamma+\sigma+1-\delta}{2}\ ; \end{matrix}\ \dfrac{u_1x_1+\cdots+u_rx_r}{1+x_1+\cdots+x_r}\right]\Big\}$$

$$= \Phi_2^{(r)}\left(\alpha_1,\ldots,\alpha_r;\ \frac{\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta}{2};\ u_1,\ldots,u_r\right)$$

where all the conditions of (7.4.16) are satisfied.

(7.4.19) $$A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\left\{{}_3F_3\left[\begin{matrix}\sigma,\alpha_1+\ldots+\alpha_r, \\ \beta+\sigma-\delta,\ \dfrac{\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta}{2},\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}\dfrac{\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta}{2}; \\ \dfrac{\alpha_1+\ldots+\alpha_r+\gamma+\sigma+1-\delta}{2};\end{matrix}\ \frac{u_1x_1+\ldots+u_rx_r}{1+x_1+\ldots+x_r}\right]\right\}$$

$$= \Phi_2^{(r)}\left(\alpha_1,\ldots,\alpha_r;\ \frac{1+\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta}{2};\ u_1,\ldots,u_r\right)$$

provided that all the conditions of (7.4.16) are satisfied.

(7.4.20) $$A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\left\{{}_3F_2\left[\begin{matrix}\sigma,\ \dfrac{\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta}{2}, \\ \dfrac{\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta}{2},\end{matrix}\right.\right.$$

$$\left.\left.\begin{matrix}\dfrac{1+\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta}{2}; \\ \dfrac{1+\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta}{2};\end{matrix}\ \frac{u_1x_1+\ldots+u_rx_r}{1+x_1+\ldots+x_r}\right]\right\}$$

$$= F_D^{(r)}(\beta+\sigma-\delta,\ \alpha_1,\ldots,\alpha_r;\alpha_1+\ldots+\alpha_r;\ u_1,\ldots,u_r)$$

valid if Re $(\delta) > 0$, Re $(\beta+\sigma-\delta) > 0$, Re $(\alpha_1+\dots+\alpha_r+\sigma+\gamma-\delta) > 0$, Re $(\alpha_j) > 0$, $j = 1,\dots,r$ and $|u_1| +\dots+|u_r| < 1$.

(7.4.21) $$A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\left\{ {}_3F_2\left[\begin{matrix} \alpha_1+\dots+\alpha_r, \sigma, \dfrac{\beta+\gamma+\sigma-\delta+\alpha_1+\dots+\alpha_r}{2} ; \\ \beta+\sigma-\delta , \dfrac{\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta}{2} ; \end{matrix} \dfrac{u_1x_1+\dots+u_rx_r}{1+x_1+\dots+x_r}\right]\right\}$$

$$= F_D^{(r)}\left(\frac{1+\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta}{2}, \alpha_1,\dots,\alpha_r ; \frac{1+\alpha_1+\dots+\alpha_r+\beta+\gamma+\sigma-\delta}{2} ; u_1,\dots,u_r\right),$$

where all conditions of (7.4.20) are satisfied.

(7.4.22) $$A^{\alpha_1,\dots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\left\{ {}_3F_2\left[\begin{matrix} \alpha_1+\dots+\alpha_r, \dfrac{\beta+\gamma+\sigma-\delta+\alpha_1+\dots+\alpha_r}{2} , \dfrac{1+\beta+\gamma+\sigma-\delta+\alpha_1+\dots+\alpha_r}{2} \\ \dfrac{\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta}{2} , \dfrac{1+\alpha_1+\dots+\alpha_r+\gamma+\sigma-\delta}{2} ; \end{matrix} \dfrac{u_1x_1+\dots+u_rx_r}{1+x_1+\dots+x_r}\right]\right\}$$

$$= F_D^{(r)}\left(\beta+\sigma-\delta, \alpha_1,\dots,\alpha_r;\sigma; u_1,\dots,u_r\right),$$

which is true under the conditions of (7.4.20),

$$(7.4.23)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{3:-;\ldots;-}_{1:1;\ldots;1}\left[\begin{matrix}[\alpha_1+\ldots+\alpha_r:1,\ldots,1],[\sigma:1,\ldots,1],\\ [\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:2,\ldots,2]:\end{matrix}\right.$$

$$\left.\begin{matrix}[\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta:2,\ldots,2]:-;\ldots;-;\\ [\beta_1:1];\ldots;[\beta_r:1];\end{matrix}\quad \frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]\}$$

$$= F_A^{(r)}(\beta+\sigma-\delta,\ \alpha_1,\ldots,\alpha_r;\ \beta_1,\ldots,\beta_r;\ u_1,\ldots,u_r),$$

provided that $\mathrm{Re}\,(\delta) > 0$, $\mathrm{Re}\,(\beta+\sigma-\delta) > 0$,

$\mathrm{Re}\,(\alpha_1+\ldots+\alpha_r + \sigma + \gamma-\delta) > 0$, $\mathrm{Re}\,(\alpha_j) > 0$, $j = 1,\ldots,r$

and $\sum_{i=1}^{r} |u_i| < 1.$

$$(7.4.24)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{2:1;\ldots;1}_{2:-;\ldots;-}\left[\begin{matrix}[\sigma:1,\ldots,1],\\ [\beta+\sigma-\delta:1,\ldots,1],\end{matrix}\right.$$

$$\begin{matrix}[\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta:2,\ldots,2]:[\beta_1:1];\ldots;[\beta_r:1];\\ [\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:2,\ldots,2]:-;\ldots;-;\end{matrix}$$

$$\left.\frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]\}$$

$$= F_B^{(r)}(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_r;\ \alpha_1+\ldots+\alpha_r;\ u_1,\ldots,u_r),$$

provided that $\mathrm{Re}\,(\delta) > 0$, $\mathrm{Re}\,(\beta+\sigma-\delta) > 0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) >$

$\mathrm{Re}\,(\alpha_j) > 0$, and $|u_j| < 1$, $j = 1,\ldots,r.$

$$(7.4.25)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{2:1;\ldots;1}_{2:-;\ldots;-}\left[\begin{array}{l}[\alpha_1+\ldots+\alpha_r:1,\ldots,1],\\ [\beta+\sigma-\delta:1,\ldots,1],\end{array}\right.$$

$$[\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta:2,\ldots,2]:[\beta_1:1];\ldots;[\beta_r:1];$$

$$[\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:2,\ldots,2]:-;\ldots;-;$$

$$\left.\frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]\}$$

$$= F_B^{(r)}(\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_r;\sigma;\ u_1,\ldots,u_r),$$

where all the conditions of (7.4.24) are satisfied.

$$(7.4.26)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{3:-;\ldots;-}_{1:2;\ldots;2}\left[\begin{array}{l}[\alpha_1+\ldots+\alpha_r:1,\ldots,1],\\ [\frac{1+\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta}{2}:1,\ldots,1]:\end{array}\right.$$

$$[\sigma:1,\ldots,1],[\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta:2,\ldots,2]:-;\ldots;-;$$

$$[\alpha_1:1],[\beta_1:1];\ldots;[\alpha_r:1],[\beta_r:1];$$

$$\left.\frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]\}$$

$$= F_C^{(r)}\left(\beta+\sigma-\delta,\frac{\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta}{2};\beta_1,\ldots,\beta_r,4u_1,\ldots,4u_r\right)$$

provided that $\mathrm{Re}\,(\delta) > 0$, $\mathrm{Re}\,(\beta+\sigma-\delta) > 0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) >$ all $\mathrm{Re}\,(\alpha_j) > 0$, $j = 1,\ldots,r$ and $\sum_{j=1}^{r} |(u_j)^{1/2}| < \frac{1}{2}$

$$(7.4.27)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{3:-;\ldots;-}_{1:2;\ldots;2}\left[\begin{array}{l}[\alpha_1+\ldots+\alpha_r:1,\ldots,1],[\sigma:1,\ldots,1]\\ [\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:2,\ldots,2]:\end{array}\right.$$

$$[\beta+\gamma+\sigma-\delta+\ldots+\alpha_1+\ldots+\alpha_r:2,\ldots,2]:-;\ldots;-;$$

$$[\alpha_1:1],[\beta_1:1];\ldots;[\alpha_r:1],[\beta_r:1];$$

$$\left.\frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]\}$$

$$=\Psi_2^{(r)}(\beta+\sigma-\delta;\beta_1,\ldots,\beta_r;u_1,\ldots,u_r)$$

valid if $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(\beta+\sigma-\delta)>0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta)>0$, all $\mathrm{Re}(\alpha_j)>0$, $j=1,\ldots,r$.

$$(7.4.28)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{3:-;\ldots;-}_{3:-;\ldots;-}\left[\begin{array}{l}[\alpha_1+\ldots+\alpha_r:1,\ldots,1],\\ [\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:2,\ldots,2],\end{array}\right.$$

$$[\sigma:1,\ldots,1],[\alpha_1+\ldots+\alpha_r+\beta+\gamma+\sigma-\delta:2,\ldots,2]:-;\ldots;-;$$

$$[\gamma:1,\ldots,1,0,\ldots,0],[\gamma':0,\ldots,0,1,\ldots,1]:-;\ldots;-;$$

$$\left.\frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]\}$$

$$={}^{(k)}_{(1)}E_D^{(r)}(\beta+\sigma-\delta,\alpha_1,\ldots,\alpha_r;\gamma,\gamma';u_1,\ldots,u_r),$$

provided that $\mathrm{Re}(\delta)>0$, $\mathrm{Re}(\beta+\sigma-\delta)>0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta)>0$, all $\mathrm{Re}(\alpha_j)>0$, $j=1,\ldots,r$ and if $|u_i|<r_i$, $i=1,\ldots,r$ then $r_1=\ldots=r_k$, $r_{k+1}=\ldots=r_r$, $r_k+r_r=1$, where $r_1,\ldots,r_r$ being the

associated radii of convergence of the series ${}^{(k)}_{(1)}E_D^{(r)}$

$$(7.4.29)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{4:-;\ldots;-}_{2:-;\ldots;-}\left[\begin{array}{l}[\sigma:1,\ldots,1],\\ [\beta+\sigma-\delta:1,\ldots,1],\end{array}\right.$$

$[\beta+\gamma+\sigma-\delta+\alpha_1+\ldots+\alpha_r:2,\ldots,2],[\gamma:1,\ldots,1,0,\ldots,1]$ ,

$[\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:2,\ldots,2]$ :

$$\left.\begin{array}{l}[\gamma':0,\ldots,0,1,\ldots,1]\ :-;\ldots;-;\\ -;\ldots;-;\end{array}\quad \frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]$$

$$= {}^{(k)}_{(2)}E_D^{(r)}\ (\gamma,\gamma',\alpha_1,\ldots,\alpha_r;\alpha_1+\ldots+\alpha_r;\ u_1,\ldots,u_r)$$

provided that Re $(\delta) > 0$, Re $(\beta+\sigma-\delta) > 0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) >$ all Re $(\alpha_j) > 0$, $j = 1,\ldots,r$ and if $|u_i| < r_i$, $i = 1,\ldots,r$ then

$r_1 = \ldots = r_k$, $r_{k+1} = \ldots = r_r$

$r_k+r_r = r_k\cdot r_r$, where $r_1,\ldots,r_r$ being the associated radii of convergence of the series ${}^{(k)}_{(2)}E_D^{(r)}$.

$$(7.4.30)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\{F^{4:-;\ldots;-}_{2:-;\ldots;-}\left[\begin{array}{l}[\alpha_1+\ldots+\alpha_r:1,\ldots,1],\\ [\beta+\sigma-\delta:1,\ldots,1],\end{array}\right.$$

$[\beta+\gamma+\sigma-\delta+\alpha_1+\ldots+\alpha_r:2,\ldots,2],[\gamma:1,\ldots,1,0,\ldots,0]$ ,

$[\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:2,\ldots,2]$ :

$$[\gamma':0,\ldots,0,1,\ldots,1]:-;\ldots;-; \quad \left.\frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]\}$$

$$-;\ldots;-;$$

$$= {}^{(k)}_{(2)}E^{(r)}_{D}\;(\gamma,\gamma',\alpha_1,\ldots,\alpha_r;\sigma;\;u_1,\ldots,u_r)$$

valid if all the conditions of (7.4.29) are satisfied

$$(7.4.31)\quad A^{\alpha_1,\ldots,\alpha_r}_{\beta,\gamma,\delta,\sigma}\;\{F^{5:-;\ldots;-}_{1:2;\ldots;2}\left[\begin{array}{l}[\alpha_1+\ldots+\alpha_r:1,\ldots,1],[\sigma:1,\ldots,1],\\ [\alpha_1+\ldots+\alpha_r+\gamma+\sigma-\delta:2,\ldots,2]:\end{array}\right.$$

$$[\beta+\gamma+\sigma-\delta+\alpha_1+\ldots+\alpha_r:2,\ldots,2],[\gamma:1,\ldots,1,0,\ldots,0],$$

$$[\alpha_1:1],\;[\beta_1:1];\ldots;$$

$$[\gamma':0,\ldots,0,1,\ldots,1]:-;\ldots;-; \quad \left.\frac{u_1x_1}{1+x_1+\ldots+x_r},\ldots,\frac{u_rx_r}{1+x_1+\ldots+x_r}\right]\}$$

$$[\alpha_r:1],\;[\beta_r:1];$$

$$= {}^{(k)}_{(1)}E^{(r)}_{C}\;(\gamma,\gamma',\beta+\sigma-\delta;\;\beta_1,\ldots,\beta_r;\;u_1,\ldots,u_r)$$

provided that $\mathrm{Re}\,(\delta) > 0$, $\mathrm{Re}\,(\beta+\sigma-\delta) > 0$, $\mathrm{Re}(\alpha_1+\ldots+\alpha_r+\sigma+\gamma-\delta) > 0$, all $\mathrm{Re}\,(\alpha_j) > 0$, $j = 1,\ldots,r$ and if $|u_i| < r_i$, $i = 1,\ldots,r$ then

$$(\sqrt{r_1}+\ldots+\sqrt{r_k})^2 + (\sqrt{r_{k+1}}+\ldots+\sqrt{r_r})^2 = 1,$$

where $r_1,\ldots,r_r$ being associated radii of convergence of the series ${}^{(k)}_{(1)}E^{(r)}_{C}$.

Further specialising the variables with parameters and applying the same techniques, we can get different dimensional Gauss' transforms of other known hypergeometric functions of different variables.

7.5. Transform B.

Consider

$$(7.5.1)\quad \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{r} (\alpha_1^i x_1 + \cdots + \alpha_r^i x_r)^{\gamma_i - 1} (1 + \alpha_1^i x_1 + \cdots + \alpha_r^i x_i)^{-\sigma_i}$$

$$_2F_1 [\beta_i, \gamma_i, \delta_i; (\alpha_1^i x_1 + \cdots + \alpha_r^i x_i)]\, dx_1 \cdots dx_r$$

$$= \frac{1}{K} \prod_{i=1}^{r} \frac{\Gamma(\delta_i)\, \Gamma(\beta_i - \delta_i + \sigma_i)\, \Gamma(\gamma_i - \delta_i + \sigma_i)}{\Gamma(\sigma_i)\, \Gamma(\beta_i + \gamma_i - \delta_i + \sigma_i)}$$

where Re $(\delta_i) > 0$, Re $(\beta_i - \delta_i + \sigma_i) > 0$, Re $(\gamma_i - \delta_i + \sigma_i) > 0$,

$$i = 1, \ldots, r \text{ and } K = \begin{vmatrix} \alpha_1' & \alpha_2' & \cdots & \alpha_r' \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_r^2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1^r & \alpha_2^r & \cdots & \alpha_r^r \end{vmatrix} \neq 0,$$

which suggests to introduce the multidimensional Gauss' transform defined by (7.1.4).

Therefore, it is easy to obtain

$$(7.5.2)\quad B\{1\} = 1.$$

(7.5.3) $B\{(1+\alpha_1' x_1+\ldots+\alpha_r' x_1)^{-\sigma_1-m_1\xi_1}\ldots(1+\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^{-\sigma_r-m_r\xi_r}$

$$= \prod_{i=1}^{r} \frac{K\,\Gamma(\sigma_i)\,\Gamma(\beta_i+\gamma_i-\delta_i+\sigma_i)\,\Gamma(\delta_i)\,\Gamma(\beta_i-\delta_i+\sigma_i+m_i\xi_i)}{\Gamma(\delta_i)\,\Gamma(\beta_i-\delta_i+\sigma_i)\,\Gamma(\gamma_i-\delta_i+\sigma_i)K\,\Gamma(\sigma_i+m_i\xi_i)}$$

$$\frac{\Gamma(\gamma_i-\delta_i+\sigma_i+m_i\xi_i)}{\Gamma(\beta_i+\gamma_i-\delta_i+\sigma_i+m_i\xi_i)}$$

$$= \prod_{i=1}^{r} \frac{(\beta_i-\delta_i+\sigma_i)_{m_i\xi_i}\,(\gamma_i-\delta_i+\sigma_i)_{m_i\xi_i}}{(\sigma_i)_{m_i\xi_i}\,(\beta_i+\gamma_i-\delta_i+\sigma_i)_{m_i\xi_i}}$$

and

(7.5.4) $B\{F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\Phi'];\ldots;\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'];\ldots;\end{matrix}\right.$

$$\left.\begin{matrix}[(b^{(r)}):\Phi^{(r)}];\\ [(d^{(r)}):\delta^{(r)}];\end{matrix}\; u_1(1+\alpha_1^1 x_1+\ldots+\alpha_r' x_r)^{-\xi_1},\ldots,u_r(1+\alpha_1^r x_1+\ldots+\alpha_r^r x_r)^{-\xi_r}\right]\}$$

$$= F^{A:B'+2;\ldots;B^{(r)}+2}_{C:D'+2;\ldots;D^{(r)}+2}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\Phi'],[\beta_1-\delta_1+\sigma_1:\xi_1]\\ [(c):\Psi',\ldots,\Psi^{(r)}]:[(d'):\delta'],[\sigma_1:\xi_1]\end{matrix}\right.$$

$[\gamma_1-\delta_1+\sigma_1:\xi_1];\ldots;[(b^{(r)}):\Phi^{(r)}],[\beta_r-\delta_r+\sigma_r:\xi_r],$

$[\beta_1+\gamma_1+\sigma_1-\delta_1:\xi_1];\ldots;[(d^{(r)}):\delta^{(r)}],[\sigma_r:\xi_r],$

$$\left.\begin{matrix} [\gamma_r-\delta_r+\sigma_r:\xi_r]\,; \\ \\ [\beta_r+\gamma_r+\sigma_r-\delta_r:\ \xi_r]\,; \end{matrix} \quad u_1,\ldots,u_r \right],$$

provided that Re $(\delta_i) > 0$, Re $(\beta_i+\sigma_i-\delta_i) > 0$, Re$(\gamma_i+\sigma_i-\delta_i) > 0$,

$$1 + \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \Phi_j^{(i)} > 0, i=1,\ldots,r.$$

$$\text{and } K = \begin{vmatrix} \alpha_1' & \alpha_2' & \ldots & \alpha_r' \\ \alpha_1^2 & \alpha_2^2 & \ldots & \alpha_r^2 \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_1^r & \alpha_2^r & \ldots & \alpha_r^r \end{vmatrix} \neq 0.$$

Now specialising the values of parameters as before we can obtain multidimensional Gauss' transform of other known multiple hypergeometric functions of Chandel [2] , Exton [3], Lauricella [4] . Similarly choosing the number of variables, we can also obtain multidimensional Gauss' transforms of multiple hypergeometric functions of different variables.

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CHAPTER VIII

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# APPLICATIONS OF MULTIPLE HYPERGEOMETRIC FUNCTIONS OF SRIVASTAVA AND DAOUST IN HEAT CONDUCTION

In this chapter we shall discuss two problems on heat conduction.

## PROBLEM I

8.1. Introduction. Singh [10] used generalized hypergeometric functions of single variables in a problem on the cooling of a heated cylinder. In this chapter we employ generalized multiple hypergeometric function of Srivastava and Daoust [12] to obtain the formal solution of fundamental differential equation of cooling of an infinitely long cylinder of a radius a, heated to the temperature $u_o = f(z)$ (z is the distance from the axis) and radiating heat into the Surrounding medium at zero temperature. From the mathematical point of view, the problem reduces to solving the fundamental partial differenti[al] equation [6,p.155 (6.7.1)].

(8.1.1) $$C_1\ \sigma\ \frac{\partial u}{\partial t} = K\ \nabla^2\ u,$$

Subject to the boundary condition.

(8.1.2) $$\left.\frac{\partial u}{\partial z} + h_1 u\right|_{z=a} = 0$$

and the initial condition

(8.1.3) $\quad u|_{t=0} = u_o = f(z),$

where the object has thermal conductivity K, heat capacity $C_1$, density $\sigma$, emmissivity $\lambda$ and $h_1 = \frac{\lambda}{K}$ .

8.2. An integral. In this section, we establish the following integral involving Srivastava and Daoust function [12], which will be used in our investigations :

(8.2.1)
$$\int_0^a z^{2\alpha-1} (a^2-z^2)^{\beta-1} J_o(\tfrac{z}{a} w_m) \, S^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}} \left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}] ; \\ [(c):\Psi',\ldots,\Psi^{(r)}] : [(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}] ; \end{matrix} \; x_1(\tfrac{z}{a})^{2\lambda_1}(1 - \tfrac{z^2}{a^2})^{\mu_1},\ldots,x_r(\tfrac{z}{a})^{2\lambda_r}(1 - \tfrac{z^2}{a^2})^{\mu_r} \right] dz$$

$$= \frac{a^{2\alpha+2\beta-2}}{2} \, S^{A+2:B';\ldots;B^{(r)};-}_{C+1:D';\ldots;D^{(r)};1} \left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(r)},0], [\alpha:\lambda_1,\ldots,\lambda_r,1] , [\beta:\mu_1,\ldots,\mu_r,0]:[(b'):\phi'];\ldots;[(b^{(r)}):\phi^{(r)}];-; \\ [(c):\Psi',\ldots,\Psi^{(r)},0] , [\alpha+\beta:\lambda_1+\mu_1,\ldots,\lambda_r+\mu_r,1],[(d'):\delta'];\ldots;[(d^{(r)}):\delta^{(r)}]; [1:1] ; \end{matrix} \; x_1,\ldots,x_r, -\frac{w_m^2}{4} \right]$$

provided that Re $(\beta) > 0$, all $\lambda_i$, $\mu_i$ are positive real numbers and

$$1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0,$$

$i = 1,\dots,r.$

8.3. Solution of the problem. The solution of (8.1.1) to be obtained is

$$(8.3.1) \quad u(z,t) = a^{2\alpha+2\beta-4} \sum_{n=1}^{\infty} \frac{J_o(\frac{z}{a} w_n)\, e^{\frac{-w_n^2 t}{a^2 B}}}{[J_1^2(w_n)+J_o^2(w_n)]}$$

$$S^{A+2:B';\dots;B^{(r)};-}_{C+1:D';\dots;D^{(r)};1} \left[ \begin{array}{l} [(a):\theta',\dots,\theta^{(r)},0],[\alpha:\lambda_1,\dots,\lambda_r,1], \\ [(c):\psi',\dots,\psi^{(r)},0],[\beta+\alpha:\lambda_1+\mu_1,\dots,\lambda_r+\mu_r,1]: \end{array} \right.$$

$$\left. \begin{array}{l} [\beta:\mu_1,\dots,\mu_r,0]:[(b'):\phi'];\dots;[(b^{(r)}):\phi^{(r)}];-; \\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}];[1:1]; \end{array} x_1,\dots,x_r,\frac{-w_n^2}{4} \right]$$

where $B = \frac{C_1 \sigma}{K}$ and all the conditions of (8.2.1) are satisfied.

Proof. The solution of (8.1.1) as given in [6,p.156, (6,7.6)] is

$$(8.3.2) \qquad u(z,t) = \sum_{n=1}^{\infty} M_n J_o(\frac{z}{a} w_n)\, e^{\frac{-w_n^2 t}{a^2 B}}$$

Due to initial condition (8.1.3), the coefficients $M_n$ may be chosen to satisfy the relation

$$(8.3.3) \qquad f(z) = \sum_{n=1}^{\infty} M_n\, J_o\,(\tfrac{z}{a}\, w_n), \quad 0 \leq z < a\,.$$

Now consider

$$(8.3.4) \qquad f(z) = z^{2\alpha-2}\,(a^2-z^2)^{\beta-1}\, S^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}] ; \\ [(c):\psi',\ldots,\psi^{(r)}] : [(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}] ;\end{matrix}\right.$$

$$\left. x_1(\tfrac{z}{a})^{2\lambda_1}\,(1-\frac{z^2}{a^2})^{\mu_1},\ldots,x_r\,(\tfrac{z}{a})^{2\lambda_r}\,(1-\frac{z^2}{a^2})^{\mu_r}\right]$$

Therefore

$$(8.3.5) \qquad z^{2\alpha-2}\,(a^2-z^2)^{\beta-1}\, S^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}] ; \\ [(c):\psi',\ldots,\psi^{(r)}] : [(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}] ;\end{matrix}\right.$$

$$\left. x_1(\tfrac{z}{a})^{2\lambda_1}\,(1-\frac{z^2}{a^2})^{\mu_1},\ldots,\ x_r(\tfrac{z}{a})^{2\lambda_r}\,(1-\frac{z^2}{a^2})^{\mu_r}\right]$$

$$= \sum_{n=1}^{\infty} M_n\, J_o\,(\tfrac{z}{a}\, w_n);\ 0 \not< z < a.$$

Multiplying both sides of (8.3.5) by $z\, J_o(\frac{z}{a}\, w_m)$ and integrating it with respect to $z$ from 0 to $a$, we get

$$(8.3.6)\quad \int_0^a z^{2\alpha-1}(a^2-z^2)^{\beta-1} J_o(\tfrac{z}{a}w_n)\, S^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}] ;\\ [(c):\psi',\ldots,\psi^{(r)}] : [(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}] ;\end{matrix}\right.$$

$$\left. x_1(\tfrac{z}{a})^{2\lambda_1}(1-\tfrac{z^2}{a^2})^{\mu_1} ;\ldots; x_r(\tfrac{z}{a})^{2\lambda_r}(1-\tfrac{z^2}{a^2})^{\mu_r}\right] dz$$

$$= \sum_{n=1}^{\infty} M_n \int_0^a z\, J_o(\tfrac{z}{a}w_n)\, J_o(\tfrac{z}{a}w_m)\, dz.$$

Now an appeal to the orthogonal property of $J_n(x)$ [1,p.130, (5.14.9)]

$$\int_0^a z\, J_\nu(\tfrac{z}{a}w_m)\, J_\nu(\tfrac{z}{a}w_n)\, dz = 0, \text{ when } m \neq n$$

$$= \frac{a^2}{2}[J_\nu'^2(w_n) + (1-\frac{\nu^2}{w_n^2})\, J_\nu^2(w_n)], \quad m = n$$

and the integral (8.2.1) with the relation

$$J_o'(x) = -J_1(x),$$

gives

$$(8.3.7)\quad M_n = \frac{a^{2\alpha+2\beta-4}}{[J_1^2(w_n)+J_o^2(w_m)]}\, S^{A+2:B';\ldots;B^{(r)};-}_{C+1:D';\ldots;D^{(r)};1}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)},0], [\alpha:\lambda_1,\ldots,\lambda_r,1],\\ [(c):\psi',\ldots,\psi^{(r)},0], [\alpha+\beta:\lambda_1+\mu_1,\ldots,\lambda_r+\mu_r,1] :\end{matrix}\right.$$

$$\left.\begin{matrix}[\beta:\mu_1,\ldots,\mu_r,0] : [(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}] ;-; \\ [(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}] ; [1:1] ;\end{matrix}\; x_1,\ldots,x_r,\frac{-w_m^2}{4}\right]$$

Now substituting the value of $M_n$ in (8.3.2), we obtain the complete solution of the problem.

8.4. <u>An expansion formula</u>. From (8.3.5) and (8.3.7) we establish the following expansion formula:

(8.4.1) $$z^{2\alpha-2}\,(a^2-z^2)^{\beta-1}\, S^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}] : [(b'):\phi'] ;\ldots; [(b^{(r)}):\phi^{(r)}] ; \\ [(c):\Psi',\ldots,\Psi^{(r)}] : [(d'):\delta'] ;\ldots; [(d^{(r)}):\delta^{(r)}] ;\end{matrix}\right.$$

$$\left. x_1\left(\frac{z}{a}\right)^{2\lambda_1}\left(1-\frac{z^2}{a^2}\right)^{\mu_1},\ldots,x_r\left(\frac{z}{a}\right)^{2\lambda_r}\left(1-\frac{z^2}{a^2}\right)^{\mu_r}\right]$$

$$= a^{2\alpha+2\beta-4}\sum_{n=1}^{\infty}\frac{J_o(\frac{z}{a}w_n)}{[J_1^2(w_n)+J_o^2(w_n)]}\, S^{A+2:B';\ldots;B^{(r)};-}_{C+1:D';\ldots;D^{(r)};1}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)},0] , [\alpha:\lambda_1,\ldots,\lambda_r,1] , [\beta:\mu_1,\ldots,\mu_r,0] : \\ [(c):\Psi',\ldots,\Psi^{(r)},0] , [\alpha+\beta:\lambda_1+\mu_1,\ldots,\lambda_r+\mu_r,1] :[(d'):\delta'];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'] ,\ldots, [(b^{(r)}):\phi^{(r)}] ;-; \\ [(d^{(r)}):\delta^{(r)}] , [1:1] ;\end{matrix}\; x_1,\ldots,x_r,\frac{-w_n^2}{4}\right]$$

provided that all the conditions of (8.2.1) are satisfied.

PROBLEM II

8.5. Introduction. Recently Singh [11] evaluated some integrals involving Kampé de Fériet function and one of them has been employed to obtain a solution of a problem in heat conduction given by Bhonsale [1]. Some expansion formulae involving above function have also been obtained. The present study is inspired by the frequent requirement of various properties of special functions which play a vital role in the study of potential theory, heat conduction and other allied problems in quantum mechanics. Appell's functions and the functions related to them have many applications in Mathematical Physics [4,7,8].

In this chapter, we evaluate certain integrals involving multiple hypergeometric function of Srivastava and Daoust [12] and their applications will be made in solving a problem on heat conduction given by Bhonsle [1] and in establishing some expansion formulae involving the above functions.

8.6. Formulae Required.

Multiplying both sides of Lebdev equation [1,(4.16.1)] by $e^{-z^2} H_{2\nu}(z)$ and using orthogonal property of Hermite polynomials [9]

$$(8.6.1) \quad \int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z)\, dz = \frac{\sqrt{\pi}\; 2^{2(\nu-\rho)}\, \Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)},$$

$\rho = 0,1,2,\ldots$

which will be useful in our investigation.

Another formula required in our investigation is due to [3]

$$(8.6.2) \qquad \int_{-1}^{1} (1-z^2)^{\rho-1} \, P_\nu^\mu(z) \, dz$$

$$= \frac{\pi \, 2^\mu \, \Gamma(\rho + \frac{\mu}{2}) \, \Gamma(\rho - \frac{\mu}{2})}{\Gamma(1+\rho+\frac{\nu}{2}) \, \Gamma(\rho - \frac{\nu}{2}) \, \Gamma(\frac{\nu}{2} + \frac{\mu}{2} + 1) \, \Gamma(-\frac{\nu}{2} - \frac{\mu}{2} + \frac{1}{2})} ,$$

$2 \operatorname{Re}(\rho) > |\operatorname{Re}(\mu)|$ .

8.7. Integrals

Making an appeal to (8.6.1) , we obtain

$$(8.7.1) \qquad \int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z) \, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}} \left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{matrix} \; x_1 z^{2\alpha_1} \ldots x_n z^{2\alpha_n} \right] dz$$

$$= \sqrt{\pi} \, 2^{2(\nu-\rho)} \, S^{A+1:B';\ldots;B^{(n)}}_{C+1:D';\ldots;D^{(n)}} \left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(n)}],[2\rho+1:2\alpha_1,\ldots,2\alpha_n]:[(b'):\phi'];\ldots; \\ [(c):\psi',\ldots,\psi^{(n)}],[1+\rho-\nu,\alpha_1,\ldots,\alpha_n]:[(d'):\delta' ;\ldots; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b^{(n)}):\Phi^{(n)}] ; \\ [(d^{(n)}):\delta^{(n)}] ; \end{matrix} \quad \frac{x_1}{2^{2\alpha_1}}, \ldots, \frac{x_n}{2^{2\alpha_n}} \right]$$

where

$$1 - \alpha_i + \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \Phi_j^{(i)} > 0,$$

$i = 1, \ldots, n$; $\rho = 0, 1, 2, \ldots,$ and $\alpha_1, \ldots, \alpha_n$ are real and positive.

Similarly an appeal to (8.6.2) shows that

$$(8.7.2) \quad \int_{-1}^{1} (1-z^2)^{\rho-1} P_\nu^\mu(z) \, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$$

$$\left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(n)}]:[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}] ; \\ [(c):\Psi',\ldots,\Psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}] ; \end{matrix} \right.$$

$$\left. x_1(1-z^2)^{\alpha_1}, \ldots, x_n(1-z^2)^{\alpha_n} \right] dz$$

$$= \frac{\pi \, 2^\mu}{\Gamma(\frac{\mu+\nu}{2}+1)\,\Gamma(\frac{1-\mu-\nu}{2})} \, S^{A+2:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}$$

$$\left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(n)}],\ [\rho+\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n],\ [\rho-\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n] : \\ [(c):\Psi',\ldots,\Psi^{(n)}],\ [\rho-\frac{\nu}{2}:\alpha_1,\ldots,\alpha_n],\ [1+\rho+\frac{\nu}{2}:\alpha_1,\ldots,\alpha_n] : \end{matrix} \right.$$

$$\left.\begin{array}{l}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{array} \quad x_1,\ldots,x_n \right]$$

provided that $2\,\mathrm{Re}\,(\rho) > |\mathrm{Re}(\mu)|$;

$$1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} > 0,$$

$i = 1,\ldots,n$ and $\alpha_1,\ldots,\alpha_n$ are real and positive.

We shall make applications of these results in our further investigations.

## 8.8. Applications to Heat Conduction.

Hermite polynomials have been utilized by Kampé de Fériet [5] in solving a heat conduction equation. He has obtained four theorems which are of the nature of existence theorems. Bhonsle [1] has also employed Hermite polynomials in solving the partial differential equation

$$(8.8.1) \qquad \frac{\partial \Phi}{\partial t} = K \frac{\partial^2 \Phi}{\partial z^2} - K\Phi z^2,$$

where $\Phi(z,t)$ tends to zero for large values of $t$ and when $|z| \to \infty$, this equation is related to the problem of heat conduction [2]

$$(8.8.2) \qquad \frac{\partial \Phi}{\partial t} = K \frac{\partial^2 \Phi}{\partial z^2} - h_1(\Phi - \Phi_0)$$

provided that $\Phi_0 = 0$ and $h_1 = Kz^2$.

The solution of (8.8.1) given by Bhonsale [1] is

$$(8.8.3) \qquad \Phi(z,t) = \sum_{r=0}^{\infty} \Phi_r \, e^{-(1+2r)Kt - \frac{z^2}{2}} H_r(z).$$

We shall consider the problem of determining a function $\Phi(z,t)$; if $t = 0$ then

$$(8.8.4) \qquad \Phi(z,0) = z^{2\rho} e^{-z^2} S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}} \left[\begin{array}{l} [(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{array} x_1 z^{2\alpha_1},\ldots,x_n z^{2\alpha_n}\right].$$

Now by (8.8.3) and (8.8.4), we have

$$(8.8.5) \qquad z^{2\rho} e^{-z^2} S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}} \left[\begin{array}{l} [(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]; \\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{array} x_1 z^{\alpha_1},\ldots,x_n z^{2\alpha_n}\right]$$

$$= \sum_{r=0}^{\infty} Q_r \, e^{-z^2/2} H_r(z).$$

Therefore

$$(8.8.6)\qquad \int_{-\infty}^{\infty} z^{2\rho}\, e^{-z^2}\, H_\mu(z)\, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$$

$$\left[\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}] : [(b'):\Phi'] ;\ldots; [(b^{(n)}):\Phi^{(n)}] ; \\ [(c):\Psi',\ldots,\Psi^{(n)}] : [(d'):\delta'] ;\ldots; [(d^{(n)}):\delta^{(n)}] ; \end{matrix}\; x_1 z^{2\alpha_1},\ldots,x_n z^{2\alpha_n}\right] dz$$

$$= \sum_{r=0}^{\infty} Q_r \int_{-\infty}^{\infty} e^{-z^2/2}\, H_r(z)\, H_\mu(z)\, dz$$

$$= Q_\mu \int_{-\infty}^{\infty} e^{-z^2/2}\, H_\mu^2(z)\, dz$$

$= (2\pi)^{1/2}\,\mu!\,Q_\mu$ {See for instance [3,p.289] }.

Hence

$$(8.8.7)\qquad Q_\mu = \frac{2^{\mu-2\rho-\frac{1}{2}}}{\mu!}\, S^{A+1:B';\ldots;B^{(n)}}_{C+1:D';\ldots;D^{(n)}}$$

$$\left[\begin{matrix} [(a):\theta',\ldots,\theta^{(n)}] , [2\rho+1:2\alpha_1,\ldots,2\alpha_n] : [(b'):\Phi'] ;\ldots; \\ [(c):\Psi',\ldots,\Psi^{(n)}] , [1+\rho-\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n] : [(d'):\delta'] ;\ldots; \end{matrix}\right.$$

$$\left.\begin{matrix} [(b^{(n)}):\Phi^{(n)}] ; \\ [(d^{(n)}):\delta^{(n)}] ; \end{matrix}\quad \frac{x_1}{2^{2\alpha_1}},\ldots,\frac{x_n}{2^{2\alpha_n}}\right].$$

Thus the solution (8.8.3) of the problem reduces to the form

$$(8.8.8) \quad \Phi(z,t) = \sum_{r=0}^{\infty} e^{-(1+2r)kt - \frac{z^2}{2}} H_r(z) \frac{2^{r-2\rho-\frac{1}{2}}}{r!}$$

$$S^{A+1:B';\ldots;B^{(n)}}_{C+1:D';\ldots;D^{(n)}}$$

$$\left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(n)}] , [2\rho+1:2\alpha_1,\ldots,2\alpha_n] : [(b'):\phi'] ;\ldots; \\ [(c):\Psi',\ldots,\Psi^{(n)}] , [1+\rho-\frac{r}{2}:\alpha_1,\ldots,\alpha_n] : [(d'):\delta'] ;\ldots; \end{matrix} \right.$$

$$\left. \begin{matrix} [(b^{(n)}):\phi^{(n)}] ; \\ [(d^{(n)}):\delta^{(n)}] ; \end{matrix} \quad \frac{x_1}{2^{2\alpha_1}} ,\ldots, \frac{x_n}{2^{2\alpha_n}} \right] ;$$

where all conditions of (8.7.1) are satisfied.

## 8.9. Expansion Formulae

In this section, we establish the following two expansion formulae :

$$(8.9.1) \quad z^{2\rho} e^{-\frac{z^2}{2}} S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$$

$$\left[ \begin{matrix} [(a):\theta',\ldots,\theta^{(n)}] : [(b'):\phi'] ; [(b^{(n)}):\phi^{(n)}] ; \\ [(c):\Psi',\ldots,\Psi^{(n)}] : [(d'):\delta'] ; [(d^{(n)}):\delta^{(n)}] ; \end{matrix} \quad x_1 z^{2\alpha_1} ,\ldots, x_n z^{2\alpha_n} \right]$$

$$= \sum_{r=0}^{\infty} \frac{2^{r-2\rho-\frac{1}{2}}}{r!} H_r(z)\, S^{A+1:B';\ldots;B^{(n)}}_{C+1:D';\ldots;D^{(n)}}$$

$$\left[\begin{array}{l} [(a):\theta',\ldots,\theta^{(n)}],\ [2\rho+1:2\alpha_1,\ldots,2\alpha_n] : [(b'):\Phi'];\ldots; \\ [(c):\Psi',\ldots,\Psi^{(n)}],\ [1+\rho-\frac{r}{2}:\alpha_1,\ldots,\alpha_n] : [(d'):\delta'];\ldots; \end{array}\right.$$

$$\left.\begin{array}{l} [(b^{(n)}):\Phi^{(n)}]; \\ [(d^{(n)}):\delta^{(n)}]; \end{array} \quad \frac{x_1}{2^{2\alpha_1}},\ldots,\frac{x_n}{2^{2\alpha_n}}\right]$$

and

$$(8.9.2) \quad (1-z^2)^{\rho-1}\, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$$

$$\left[\begin{array}{l} [(a):\theta',\ldots,\theta^{(n)}] : [(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [(c):\Psi',\ldots,\Psi^{(n)}] : [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]; \end{array} \quad x_1(1-z^2)^{\alpha_1},\ldots,x_n(1-z^2)^{\alpha_n}\right]$$

$$= \sum_{\nu=0}^{\infty} \frac{\pi\, 2^{\mu-1}\,(2\nu+1)\,(\nu-\mu)!}{(\mu+\nu)!\,\Gamma(\frac{\mu+\nu}{2}+1)\,\Gamma(\frac{1-\mu-\nu}{2})} P^{\mu}_{\nu}(z)$$

$$S^{A+2:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}$$

$$\left[\begin{array}{l}[(a):\theta',\ldots,\theta^{(n)}],[\rho+\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n],[\rho-\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n]: \\ [(c):\Psi',\ldots,\Psi^{(n)}],[1+\rho+\frac{\nu}{2}:\alpha_1,\ldots,\alpha_n],[\rho-\frac{\nu}{2}:\alpha_1,\ldots,\alpha_n]:\end{array}\right.$$

$$\left.\begin{array}{l}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{array} \quad x_1,\ldots,x_n\right].$$

Proof :- The expansion (8.9.1) can be directly established by (8.8.5) and (8.8.7).

To prove (8.9.2), let

(8.9.3) $$(1-z^2)^{\rho-1}\, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$$

$$\left[\begin{array}{l}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}]; \\ [(c):\Psi',\ldots,\Psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{array}\right.$$

$$\left. x_1(1-z^2)^{\alpha_1},\ldots,x_n(1-z^2)^{\alpha_n}\right]$$

$$=\sum_{r=0}^{\infty} A_r\, P_r^{\mu}(z)$$

Therefore,

$$\int_{-1}^{1}(1-z^2)^{\rho-1}\, P_\nu^\mu(z)\, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}] : [(b'):\Phi'] ;\ldots; [(b^{(n)}):\Phi^{(n)}] ; \\ [(c):\Psi',\ldots,\Psi^{(n)}] : [(d'):\delta'] ;\ldots; [(d^{(n)}):\delta^{(n)}] ;\end{matrix}\right.$$

$$\left. x_1(1-z^2)^{\alpha_1},\ldots,x_n(1-z^2)^{\alpha_n} \right] dz$$

$$= \sum_{r=0}^{\infty} A_r \int_{-1}^{1} P_\nu^\mu(z)\ P_r^\mu(z)\ dz$$

$$= A_\nu \int_{-1}^{1} [P_\nu^\mu(z)]^2\ dz$$

$$= A_\nu \frac{2\ (\mu+\nu)!}{(2\nu+1)(\nu-\mu)!}$$ {See for instance [3,p.278] },

which gives

$$(8.9.4)\quad A_\nu = \frac{(2\nu+1)\ (\nu-\mu)!\,\pi\ 2^{\mu-1}}{(\mu+\nu)!\,\Gamma(\frac{\mu+\nu}{2}+1)\ \Gamma(\frac{1-\mu-\nu}{2})}$$

$$S^{A+2:B';\ldots;B^{(n)}}_{C+2:D';\ldots;D^{(n)}}$$

$$\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\ [\rho+\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n],\ [\rho-\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n] : \\ [(c):\Psi',\ldots,\Psi^{(n)}],\ [1+\rho+\frac{\nu}{2}:\alpha_1,\ldots,\alpha_n],\ [\rho-\frac{\nu}{2}:\alpha_1,\ldots,\alpha_n] :\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots; [(b^{(n)}):\Phi^{(n)}] ; \\ [(d'):\delta'];\ldots; [(d^{(n)}):\delta^{(n)}] ;\end{matrix}\quad x_1,\ldots,x_n \right]$$

Thus an appeal to (8.9.3) and (8.9.4) gives the proof of (8.9.2).

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